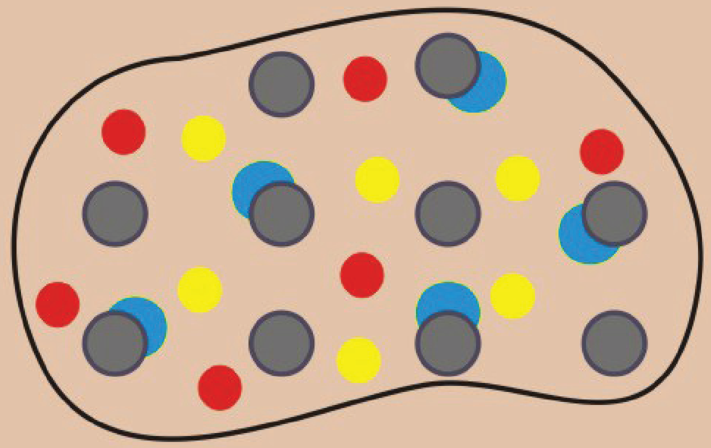
Homogenization of a System of Nonlinear Multi-Species Diffusion-Reaction Equations in an $H^{1,p}$ Setting



Hari Shankar Mahato

Homogenization of a System of Nonlinear Multi-Species Diffusion-Reaction Equations in an $H^{1,p}$ Setting

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Dissertation
zur Erlangung des Grades eines Doktors der Naturwissenschaften
- Dr. rer. nat.-

Vorgelegt im Fachbereich 3 (Mathematik & Informatik)
der Universität Bremen
im März 2013

Datum des Promotionskolloquiums: 17.05.2013

Gutachter: Prof. Dr. habil. Michael Böhm (Universität Bremen)
Prof. Dr. habil. Peter Knabner (Universität Erlangen-Nürnberg)

Dedicated in the loving memory of my mother
who could not live longer to see me concluding this work
but she always believed that
I have the courage to face any obstacle.

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Abstract

The processes of chemical transport in porous media are extensively studied in the fields of applied mathematics, material science, chemical engineering etc. A porous medium (e.g. concrete, soil, rocks, reservoir etc.) is a multiscale material/medium where the heterogeneities present in the medium are characterized by the *micro scale* and the global behaviors of the medium are observed by the *macro scale*. The upscaling from the micro scale to the macro scale can be done via averaging methods.

The transport process in a porous medium is a complex phenomena. In this thesis, the heterogeneities inside a porous medium are assumed to be periodically distributed and diffusion-reaction of a finite number of chemical species are investigated. Two different models are proposed in this work. In model M1, diffusion-reaction of mobile chemical species are considered. The chemical processes are modeled via mass action kinetics and the modeling leads to a system of multi-species diffusion-reaction equations (nonlinear partial differential equations) at the micro scale. For this system of equations, existence of a unique positive global weak solution is proved by the help of a *Lyapunov functional* and *Schaefer's fixed point theorem*. The upscaled model of this system is obtained using *periodic homogenization* which is an averaging method.

In model M2, we consider diffusion-advection-reaction of two different types of mobile species (type I and type II). The type II species are supplied via dissolution process due to the presence of immobile species on the surface of the solid parts. The presence of mobile and the immobile species make the model complex and the modeling yields a coupled system of nonlinear partial differential equations. The existence of a unique positive global weak solution of this complex system is shown. Finally, with the help of periodic homogenization, model M2 is upscaled from the micro scale to the macro scale.

Numerical simulations are conducted for both models separately. For the purpose of illustration, we restrict ourselves to relatively simple 2-dimensional situations. For models M1 and M2, simulation results at the micro scale and at the macro scale are compared.

Acknowledgments

I am sincerely and heartily thankful to my supervisor Prof. Dr. Michael Böhm for his great support during my dissertation. Without his constant guidance and care, both academically and personally, this work could not be possible. I would also like to thank Prof. Dr. Peter Knabner for his interest in my work, for his helpful comments and suggestions and for agreeing to be the second examiner of my thesis.

I also owe earnest thankfulness for the research facilities provided by my working group ”Mathematical Modeling and Partial Differential Equations” and by the ”Center of Industrial Mathematics (ZeTeM)”, both situated at the University of Bremen. Special thanks go to Simone Bökenheide, Dr. Sören Boettcher, Dr. Sören Dobberschütz, Michael Eden, Martin Höpker, Nils Hendrik Kröger, Dr. Sebastian Meier, Daniel Scholten, Dr. Jonathan M. Urquizo and PD Dr. Michael Wolff for proof reading of the thesis. I would also like to thank Dr. Joachim Rehberg, Dr. Marita Thomas and Sina Reichelt from WIAS Berlin, and Dr. Maria Neuss-Radu, Dr. Serge Kräutle and Nadja Ray from the University of Erlangen-Nürnberg for their helpful comments. And last but not least, I would like to thank Mrs. Julitta von Deetzen for taking care of all the academic matters.

I acknowledge the financial support of DFG (German Research Foundation) and University of Bremen with the help of whom I conducted this research successfully.

I highly appreciate the wholehearted support of my family and friends throughout these four years. Their constant belief has provided me the strength to accomplish this work.

May 17, 2013 *Hari Shankar Mahato*
Bremen, Germany

List of Mathematical Notations

$\mathbb{R}$ (resp. $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{K}$)	set of real numbers (resp. integers, complex, scalar)
$\mathbb{R}^+$ (resp. $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{N}$)	set of positive real numbers (resp. positive integers negative integers, natural numbers)
$\mathbb{R}_0^+$ (resp. $\mathbb{Z}_0^+$, $\mathbb{Z}_0^-$, $\mathbb{N}_0$)	$\mathbb{R}^+ \cup \{0\}$ (resp. $\mathbb{Z}^+ \cup \{0\}$, $\mathbb{Z}^- \cup \{0\}$, $\mathbb{N}^+ \cup \{0\}$)
$A - B$	$\{x \in A \,:\, x \notin B\}$
$(\ ,\ )$ (resp. $[\ ,\ ]$, or $[\ ,\ )$, $(\ ,\ ]$)	open interval (resp. closed, or semi-open intervals)
$\|\|.\|\|_X$	norm on the linear space X, see page 18
X^*	dual space of X
$\mathcal{L}(X,Y)$	set of all continuous linear operators from X to Y
$(\,.\,,\,.\,)_H := (\,.\,,\,.\,)$	inner product on a Hilbert space H
$\langle\,.\,,\,.\,\rangle_{X\times X^*}$	duality paring between X and X^*
I, I_1, I_2	positive integers (number of chemical species)
X^I	$\underbrace{X \times X \times \ldots \times X}_{I-times}$
$\|\|\|.\|\|\|_{X^I}$	norm on the vector-valued space X^I, see page 19
$\langle\,.\,,\,.\,\rangle_I$	Euclidean inner product in $\mathbb{R}^I$
$\langle\,.\,,\,.\,\rangle_{X^I\times[X^*]^I}$	duality paring between X^I and $[X^*]^I$
$\|.\|_I$	Euclidean norm in $\mathbb{R}^I$
p	a real number in $(1,\infty)$
q	$(\,p\,)'$ = dual index of q, i.e., $\dfrac{1}{p}+\dfrac{1}{q}=1$
$\subset$	subset
$\mapsto$	maps to
$\to$	strong convergence
$\rightharpoonup$	weak convergence
$\overset{w^*}{\rightharpoonup}$	weak-star convergence
$\overset{d}{\subset}$	dense subset
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
$\overset{2}{\rightharpoonup}$	two-scale convergence
$\Longrightarrow$	implies
$\Longleftrightarrow$	if and only if
$\rightleftharpoons$	reversible reaction
δ_{jk}	Kronecker delta

$d\sigma_y$	surface measure on Γ, see page 15
$d\sigma_x$	surface measure on Γ_ε, see page 15
∇, ∇_x, ∇_y	gradient operator; $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)$, where $x = (x_1, x_2, ..., x_n)$
div, div_x, div_y	divergence of a vector function
Ω	bounded domain in $\mathbb{R}^n$
S	$[0,T)$, the time interval
$\frac{\partial u}{\partial t}$	time derivative of u with respect to t in the distributional sense
$\partial_j u$	$\frac{\partial u}{\partial x_j}$, where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$
α	a multi-index such that $\alpha_1 + \alpha_2 + ... + \alpha_n = \lvert\alpha\rvert$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}_+^n$
$D^\alpha u$	$\frac{\partial^{\lvert\alpha\rvert} u}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_n}^{\alpha_n}}$
$(\,.\,,\,.\,)_{\theta,p}$	real-interpolation space, where $0 < \theta < 1$
$[\,.\,,\,.\,]_\theta$	complex interpolation space, where $0 < \theta < 1$
$C^k(\bar{\Omega})$	space of all k-times continuously differentiable function on $\bar{\Omega}$, where $k \in \mathbb{N}$, see page 18
$C^\gamma(\bar{\Omega})$	Hölder space, where $0 < \gamma \leq 1$, see page 18
$L^p(\Omega)$	equivalence class of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\lvert u(.)\rvert^p$ is Lebesgue integrable, see page 18
$L^\infty(\Omega)$	equivalence class of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\operatorname{ess\,sup}_{x\in\Omega} \lvert u(x)\rvert < \infty$, see page 18
$L^p_+(\Omega)$	$\{u \in L^p(\Omega) : u \geq 0 \text{ a.e.}\}$, where $1 \leq p \leq \infty$
$H^{1,p}(\Omega)$	space of locally summable functions $u : \Omega \to \mathbb{R}$ such that for every multiindex α with $\lvert\alpha\rvert \leq 1$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$, see page 18
$H^{s,p}(\Omega)$	fractional order Sobolev/Sobolev-Slobodecki space, where $s \in \mathbb{R}_0^+$, see page 19
F	Bochner space of type $H^{1,p}((0,T); H^{1,q}(\Omega)^*) \cap L^p((0,T); H^{1,p}(\Omega))$, see page 19
$\mathcal{F}_p^u$, $\mathcal{G}_p^v$, $\mathcal{H}_p^w$, $\mathcal{M}_\infty^z$	see pages 21 and 21
$\mathcal{X}_p^u$, $\mathcal{X}_p^v$, $\mathcal{X}_p^w$	see page 21
$\mathcal{F}_\varepsilon^u$, $\mathcal{G}_\varepsilon^v$, $\mathcal{H}_\varepsilon^w$, $\mathcal{M}_\varepsilon^z$	see page 21
$\mathcal{X}_{p_\varepsilon}^u$, $\mathcal{X}_{p_\varepsilon}^v$, $\mathcal{X}_{p_\varepsilon}^w$	see page 21
C, C_i	generic nonnegative constants but may be different at different steps of the inequality

List of Modeling Notations

ε	scale parameter, see page 14
Y	a representative cell in $\mathbb{R}^n$, see page 13
Y^p	pore space in Y, see page 13
Y^s	solid part in Y, see page 13
Γ	∂Y^s, i.e., boundary of Y^s, see page 13
$\vec{n}$	unit outward drawn normal on the boundary
Ω	a bounded domain/porous medium in $\mathbb{R}^n$
Ω^p	pore space available for fluid in Ω, see page 8
Ω^s	union of the solid parts in Ω, see page 8
Γ^*	boundary of Ω^s, see page 8
$\partial\Omega$	outer boundary of Ω
$\partial\Omega^p$	$\partial\Omega \cup \Gamma^*$, see page 8
$\partial\Omega_{in}$	inflow boundary, see page 12
$\partial\Omega_{out}$	outflow boundary, see page 12
Ω^p_ε	pore space scaled by ε, see page 14
Ω^s_ε	solid parts scaled by ε, see page 14
Γ_ε	union of boundaries of the solid parts scaled by ε, see page 14
$\partial\Omega^p_\varepsilon$	$\partial\Omega \cup \Gamma_\varepsilon$
φ	porosity constant
D	diffusion coefficient
k^f_j	forward reaction rate factor in the *j-th* reaction
k^b_j	backward reaction rate factor in the *j-th* reaction
k_d	dissolution coefficient
P	positive definite diffusive tensor, see page 58
ρ	density of the fluid
$\vec{q}$	fluid velocity
ψ	positive signum function, see page 11
χ_M	characteristic/indicator function of a set M

List of Abbreviations

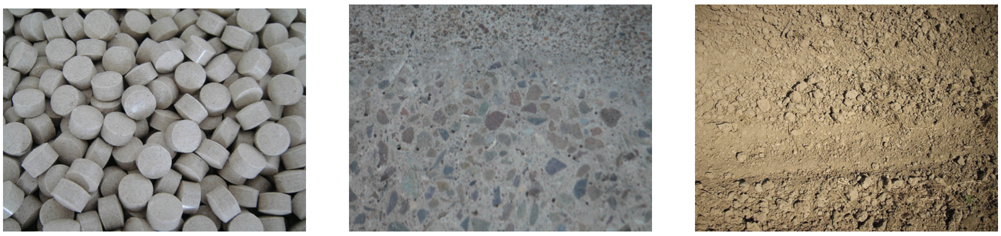
BC (resp. IC)	boundary condition (resp. initial condition)
r.h.s.	right hand side
l.h.s.	left hand side
w.r.t.	with respect to
b.v.p.	boundary value problem
⧫	end of the proof
$\because$	because/since
$\therefore$	therefore

List of Figures

Chapter 1

Introduction

Several physical problems in the fields of physics, chemistry, biology and engineering sciences are governed by *diffusion-reaction equations*. One of the important phenomena that can be explained with the help of these equations is *chemical transport in a porous medium* (e.g. soil, rock, concrete, pellets etc., see figure 1.1.1). The aim of this thesis is to investigate the transport processes of mobile and immobile chemical species present inside a porous medium.



Pellets Concrete Soil

Figure 1.1.1: Examples of a porous medium.[1]

In general, a porous medium has a complex geometry. It is a heterogeneous medium composed of pore space and union of solid parts (see figure 1.1.2), where the heterogeneities are much smaller compared to the size of the medium. Thus in order to analyze the processes which take place inside the medium, one needs to consider the *microscopic* and the *macroscopic* description of the medium. The size of the microscopic scale can vary from nanometer to micrometer and it is appropriate for describing the heterogeneities of the medium, however, it is not suitable for numerical simulations. On the other hand, the size of the macroscopic scale can vary from meter to kilometer or even larger and the macroscopic description of the medium fits well for numerical computations. Thus, to study the bulk (global) behaviors of a material, one *upscales* a mathematical model (in this thesis it is given by partial differential equations) from the micro scale to the macro scale.

In this thesis, two different models are proposed at the micro scale and the upscaled models (models at the macro scale) are obtained by periodic homogenization. Periodic homogenization refers to an averaging method in which the distribution of the solid pieces comprising the solid part (see sections 2.4 and 2.5.1) in the porous medium is periodic (cf. figure 2.5.4). The periodicity assumption of solid parts in the porous medium is used by many authors for homogenization (cf. [ADH96], [ADH90], [CD99], [Cla98], [ACP08], [HJ91], [Pet06] and references therein). In reality such a distribution of solid parts is very rarely met, however, the assumption of periodicity can be relaxed (cf. [Pet06], [Mei08], [Fat13] etc.)

The transport processes in porous media, for example in soil, have been extensively studied in last decades and it has drawn the attention of geologists, hydrologists, math-

[1] These images are taken from the website http://purechemicals.co.uk/news/tag/benzo-fury-pellets/ and http://commons.wikimedia.org/wiki/Main_Page.

ematicians and others (cf. [BB90], [vDP04], [Kna91], [Kna86], [Krä08], [Log01], [Rub83], [WR87] etc.). Recently Kräutle has shown, on the macroscopic level, the existence and uniqueness of the global solution in $[H^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);H^{2,p}(\Omega))]^I$ of a system of diffusion-reaction equations for the multi-species reactive transport problem in a porous medium, where Ω is the given porous medium, I is the number of chemical species and $p > n+1$ (cf. [Krä08], [Krä11]). With the help of a *Lyapunov functional*, he obtained some *a-priori* estimates (global in time) and showed the existence of a unique solution on the time interval $[0,T)$ for any $T > 0$. However, to our knowledge, it seems that this idea has not been excavated to its full strength when the solution $u(t)$ has derivative only up to the first order, i.e., if only $u(t) \in H^{1,p}(\Omega)$. In the first part of this work, we also consider diffusion-reaction of a finite number of chemical species[2]. Since our porous medium is heterogeneous, we consider the system of diffusion-reaction equations at the micro scale and we prove the existence of a unique positive global weak solution in $[H^{1,p}((0,T);H^{1,q}(\Omega_\varepsilon^p)^*) \cap L^p((0,T);H^{1,p}(\Omega_\varepsilon^p))]^I$ for $p > n+2$, (see section 2.5.1 for the definition of Ω_ε^p). We upscale the models governed by nonlinear partial differential equations from the micro to the macro scale using two-scale convergence and periodic unfolding (see sections 3.5 and 3.6). In the second part of this thesis, we investigate a rather complex model where we incorporate the previous model with dissolution which takes place on the surface of the solid parts (see page 10).

In this work, we will consider the following type of a porous medium:

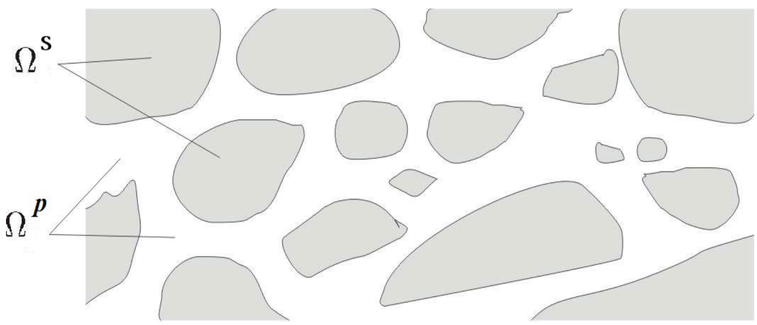


Figure 1.1.2: A typical porous medium with solid parts Ω^s and pore space Ω^p.

The transport processes take place in the pore space. In this work the pore space is assumed to be connected whereas the solid parts are considered as disconnected. It is also possible that the species present on the surface of the solid parts react with the species present in the fluid via dissolution or precipitation. This will not only lead to an extra source term but may also affect the size of the solid parts in the medium (cf. [Pet06], e.g.).

We cite some examples from the literature in which chemical transport in a porous medium has been investigated. The carbonation inside the concrete affects its durability and longevity. The authors in [SS98], [Pet06], [Mun06], [MPM+07], [MPMB07], [Mei08], [MB09b], [MB09a] (and references therein) have proposed appropriate mathematical models for the concrete carbonation and investigated the reactions associated with it. Sulfuric acid attack in sewer pipes made of concrete is studied in [BJDR98], [FAZM11], [FM12]. In [NRK08], authors have discussed the dynamics of hematopoietic stem cells (HSCs). The processes of dissolution and precipitation have been examined by many authors in the context of porous media, for example see [KvDH95], [Kna86], [vDP04] etc.

[2] The reaction rates (given by mass action law) are of the form (2.4.7) which is motivated from the work of Kräutle in [Krä08].

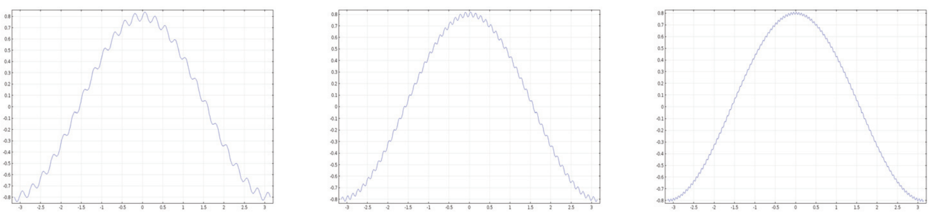
1.1 Periodic Homogenization

The goal of homogenization theory is to give a macroscopic description of a material body or of a medium which is microscopically heterogeneous, i.e., the heterogeneous body is replaced by a homogeneous body which is considered as an approximation to the heteregeneous body so that the physical properties associated with the body can be examined. Mathematically speaking, homogenization theory gives the convergence of the solutions of a given b.v.p. which has highly oscillating coefficients to the solution of a limit b.v.p. which is a good approximation of the original b.v.p., i.e., the limit b.v.p. is simpler and does not involve highly oscillating coefficients. For example, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that the heterogeneities in Ω are very small and periodically distributed. Let $\varepsilon > 0$ be the scale parameter representing the periodicity. Consider the following b.v.p.

$$L^{\varepsilon}\, u_{\varepsilon}(x) \;:=\; -\nabla\cdot(D^{\varepsilon}(x)\nabla u_{\varepsilon}(x)) \;=\; f(x) \qquad \text{in } \Omega \tag{1.1.1}$$

$$u_{\varepsilon}(x) \;=\; 0 \qquad \text{on } \partial\Omega, \tag{1.1.2}$$

where $D^{\varepsilon}(x) = D(x, \frac{x}{\varepsilon})$ for a.e. $x \in \Omega$ and D^{ε} is periodic w.r.t. $\frac{x}{\varepsilon}$ (cf. [CD99], [Hor97]). Here x is the macroscopic variable and $\frac{x}{\varepsilon}$ is the microscopic variable. To illustrate the ideas, consider $\Omega = (-\pi, \pi)$ and $D^{\varepsilon}(x) = 0.8\,\cos(x) + \varepsilon\,\sin(\frac{x}{\varepsilon})$. For different values of ε, the graphs of D^{ε} are plotted in figure 1.1.3.



For $\varepsilon = \frac{1}{25}$ For $\varepsilon = \frac{1}{50}$ For $\varepsilon = \frac{1}{100}$

Figure 1.1.3: Graph of D^{ε} for different ε.

The variation at the macro scale is given by the part $0.8\,\cos(x)$ and the oscillation at the micro scale is described by $\varepsilon \sin(\frac{x}{\varepsilon})$. By letting $\varepsilon \to 0$, we make the oscillations smaller and smaller. The numerical simulation of the model (1.1.1)-(1.1.2) is difficult due to the micro oscillations. Thus we are interested to obtain a homogenized b.v.p. (see chapter 4) which contains an averaged effect of the micro oscillations instead of involving it explicitly in the problem. Let us denote this b.v.p. by

$$L\, u(x) \;=\; -\nabla\cdot(\bar{D}\nabla u(x)) \;=\; f(x) \qquad \text{in } \Omega \tag{1.1.3}$$

$$u(x) \;=\; 0 \qquad \text{on } \partial\Omega, \tag{1.1.4}$$

where $\bar{D}$ is the "*averaged coefficient*" (see equation 4.1.101, e.g.). The homogenized equation (1.1.3)-(1.1.4) is better suitable for numerical simulations and the solution of (1.1.3)-(1.1.4) is an approximation to the solution of (1.1.1)-(1.1.2). However, the convergence of u_{ε} as $\varepsilon \to 0$ needs to be established.

To obtain the homogenized b.v.p. (i.e., to understand the convergence as $\varepsilon \to 0$), several methods have been developed:

- The first method is *asymptotic expansion* (cf. the book of A. Bensoussan, J.L. Lions and G. Papanicolaou [BLP78]). We assume that our unknown function u_{ε} has an asymptotic expansion of the form

$$u_{\varepsilon}(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots, \tag{1.1.5}$$

where the functions u_i for all i depend on x and $\frac{x}{\varepsilon}$, and are periodic w.r.t. the microscopic variable $\frac{x}{\varepsilon}$. Substituting the expansion (1.1.5) in (1.1.1) and comparing different powers of ε, one can obtain the homogenized b.v.p. This method is, however, only formal and does not give any mathematical proof of convergence.

- A mathematical proof of convergence of u_ε can be given by *oscillating test function method* developed by Tartar (see chapter 8 in [CD99]). However, for complex problems this method is not suitable.
- The notion of *two-scale convergence* has been introduced by Nguetseng (cf. [Ngu89]) and later on developed by Allaire (cf. [All92]). This method is suitable for studying the problems of the type above (see section 3.5 for definition and theorems).
- The recently developed *periodic unfolding method* by Cioranescu, Damlamian and Griso (cf. [CDG02]) has also become a very efficient tool to deal with the problems described above (cf. section 3.6 for definition and theorems). It is suitable for dealing with the nonlinear boundary value problems.

1.2 Outline of the Thesis

This thesis contains six chapters followed by an appendix. After the introductory chapter (chapter 1), we present diffusion-reaction models in chapter 2. Some mathematical tools have been collected in chapter 3. The analysis of models is done in chapter 4 and numerical simulations of models are given in chapter 5. We summarize this work in chapter 6 followed by an appendix.

In chapter 2, we start with a brief discussion on different types of fluxes in section 2.1. A very short illustration of reaction rates is given in section 2.2. We familiarize with the notions of dissolution and precipitation in section 2.3. In section 2.4, two types of diffusion-reaction models (M1 and M2, see sections 2.4.1 and 2.4.2) are introduced. The periodic scaling of the domain Ω (given porous medium) is shown in section 2.5 and we conclude chapter 2 by deriving diffusion-reaction models at the micro scale in sections 2.5.1 and 2.5.2.

In chapter 3, we collect some mathematical tools to analyze the models M1 and M2 respectively. In section 3.1, several function spaces are introduced followed by some embedding theorems and the weak formulation of models M1 and M2 at the micro scale. The concept of *maximal parabolic regularity* is given in section 3.3. Some important theorems are derived at the micro scale in section 3.4. The notions of *two-scale convergence* and *periodic unfolding* are given in sections 3.5 and 3.6 respectively.

Chapter 4 is the main body of this work. Model M1 is considered in section 4.1. In section 4.1.1, existence of a unique positive global weak solution of model M1 is shown by the help of a *Lyapunov functional* (see section 4.1.1.2), *Schaefer's fixed point theorem* (cf. theorem B.1) and the linear theory of evolution equations involving maximal regularity (cf. theorem 3.3.1). Some *a-priori* estimates of the solution of model M1 are obtained in sections 4.1.2.1 and 4.1.2.2. The homogenization of model M1 is conducted in section 4.1.2.3.

Model M2 is investigated in section 4.2. The global existence and uniqueness of a positive weak solution of M2 is proved in section 4.2.1. Some *a-priori* estimates of the solution of model M2 are obtained in sections 4.2.2.1 and 4.2.2.2. The homogenized model for model M2 is achieved in section 4.2.2.3.

In chapter 5, numerical simulations are performed. In section 5.1, simulations for model M1 at the micro scale and at the macro scale are shown. We conclude this chapter with the numerical computations for model M2 in section 5.2.

A short summary and outlook of this thesis are given in chapter 6. The appendix contains two sections. In section A, a few elementary inequalities are collected. Some classical theorems on Sobolev spaces are listed in section B.

Chapter 2

The Model

In this chapter, we introduce two different models: Model M1 and Model M2. In model M1, we consider only diffusion and reaction of chemical species inside the pore space. In this case, the species are transported via diffusion. For model M2, we consider diffusion, reaction and advection of chemical species. Here the species are transported via both advection and diffusion. The dissolution process occurs on the surface of the solid parts. We begin with the diffusion-advection equation in section 2.1. In section 2.2, we give a short description of reaction rates. Section 2.3 deals with precipitation and dissolution. The derivations of models M1 and M2 are shown in sections 2.4.1 and 2.4.2 respectively. We conclude this chapter by obtaining the settings for M1 and M2 at the microscopic scale in sections 2.5.2 and 2.5.3 respectively.

2.1 Diffusion-Advection Equation

Let $\Omega \subset \mathbb{R}^n$ be the given porous medium with sufficiently smooth boundary $\partial\Omega$. Suppose that $u(t,x)$ denotes the concentration of a chemical species, A, present in the fluid and $Q(t,x)$ is the flux, i.e., rate per unit area of the amount of species entering or leaving the domain through the boundary. Also, f denotes the rate per unit volume by which the species is either consumed or produced. Then the *diffusion-advection equation* for A is given by

$$\boxed{\varphi\frac{\partial}{\partial t}u - \nabla\cdot Q \;=\; f,} \tag{2.1.1}$$

where $\varphi \in (0,1)$ is the porosity of the medium (for derivation of (2.1.1) see [Log01]). We focus on the flux Q. In homogenization one considers two modes of transportation in the pore space. The first one is ***advection*** in which the substance is carried from one place to another by the bulk motion of the fluid present in the medium. The advective flux is given as

$$Q_{adv} \;=\; \vec{q}u, \tag{2.1.2}$$

where $\vec{q}$ is the fluid velocity. The another process by which mass can be transported is diffusion. In this work we assume the diffusive flux Q_{diff} to be given by ***Fick's law***, i.e.,

$$Q_{diff} := \text{diffusive flux} \;=\; -D_{diff}\nabla_x u, \tag{2.1.3}$$

where D_{diff} is a positive definite symmetric matrix. Later on we restrict ourselves to the case of scalar D_{diff}.

2.2 Reaction Rates

In a chemical reaction, a chemical species can either be consumed or produced. This leads us to introduce two types of reaction rates: the rate of consumption if the species is consumed and the rate of production if the species is produced. For example, let us consider N number of chemical species be involved in J number of reactions which are given as

$$\tau_{1j}X_1+\tau_{2j}X_2+\ldots+\tau_{Nj}X_N \;\rightarrow\; \nu_{1j}Y_1+\nu_{2j}Y_2+\ldots+\nu_{Nj}Y_N, \tag{2.2.1}$$

where X_i and Y_i denote the chemical species, and $-\tau_{ij}$ and ν_{ij} are the *stoichiometric coefficients* for $1\le i\le N$ and $1\le j\le J$. The rate of consumption of X_i is given by (in this setting by mass action law)

$$R_{C_{X_i}} \;=\; \sum_{j=1}^{J}-\tau_{ij}k_j\prod_{i=1}^{N}u_i^{\tau_{ij}} \;=\; -\sum_{j=1}^{J}\tau_{ij}k_j\prod_{i=1}^{N}u_i^{\tau_{ij}},$$

where k_j is the reaction rate factor. Similarly the rate of production of Y_i is given as $R_{P_{Y_i}}=\sum_{j=1}^{J}k_j\nu_{ij}\prod_{i=1}^{N}u_i^{\tau_{ij}}$. If there is no confusion, from here and on we simply prefer the term reaction rate for both the rate of consumption and the rate of production. The reactions of type (2.2.1) are called the *irreversible reactions*. Now we introduce the reversible reactions.

A *reversible reaction* is a reaction in which reactants react to form products called the *forward reaction* and products react to give the reactants back called the *backward reaction*. When the reversible reactions reach equilibrium, it means that the reaction rates are not zero but they proceed with equal rate. For example, let us consider the following reversible reaction

$$\tau_{1j}X_1+\tau_{2j}X_2+\ldots+\tau_{Nj}X_N \;\rightleftharpoons\; \nu_{1j}Y_1+\nu_{2j}Y_2+\ldots+\nu_{Nj}Y_N, 1\le j\le J, \tag{2.2.2}$$

where X_i and Y_i denote the chemical species, and $-\tau_{ij}$ and ν_{ij} are the stoichiometric coefficients for $1\le i\le N$ and $1\le j\le J$. Let u_i and v_i denote the concentrations of X_i and Y_i respectively. Then the reaction rate of the species X_i (by mass action law) is given as

$$R_{X_i}(u) \;=\; \sum_{j=1}^{J}(-\tau_{ij})R_j(u) \;=\; -\sum_{j=1}^{J}\tau_{ij}(R_j^f(u)-R_j^b(u)), \tag{2.2.3}$$

where

$$R_j^f(u) \;=\; \text{forward reaction rate} \;=\; k_j^f\prod_{m=1}^{N}u_m^{\tau_{mj}} \text{ and} \tag{2.2.4}$$

$$R_j^b(u) \;=\; \text{backward reaction rate} \;=\; k_j^b\prod_{m=1}^{N}v_m^{\nu_{mj}}, \tag{2.2.5}$$

where $k_j^f,\ k_j^b>0$ are the forward and backward reaction rate factors. Similarly, we can express the reaction rate for Y_i as well for all $1\le i\le N$. We note that the expression for reaction rate in (2.2.3) is motivated from the work of Kräutle (cf. [Krä08]).

2.3 Dissolution and Precipitation

Crystal (immobile species) dissolution is a process in which a solid substance solubilizes in a given solvent, i.e., the mass transfer from the surface of the solid parts to the liquid phase. Precipitation or adsorption is the reverse process of dissolution. When a chemical solution, containing a substance, is supersaturated or the crystals of this substance are present in the solution, precipitation occurs. Following the notion of Knabner and van Duijn (cf. [KvD96]), let c_1 and c_2 be the concentrations of two chemical species M_1 and M_2 present in the pore space. Let c_{12} be the concentration of an immobile species M_{12} attached to the solid parts. We assume that n molecules of M_1 and m molecules of M_2 precipitate to give one molecule of M_{12}. The reverse reaction of dissolution is also possible, i.e.,

$$nM_1 + mM_2 \rightleftharpoons M_{12}, \tag{2.3.1}$$

then by mass action law the rate of precipitation R_p is given by

$$R_p = k_p c_1^n c_2^m, \tag{2.3.2}$$

where k_p is the precipitation rate constant. We assume that the dissolution rate R_d is constant in the presence of immobile species on Γ^* and has to be such that in the absence of immobile species the overall rate is zero. To achieve this, we set $R_d \in k_d\,\psi(c_{12})$, where $k_d > 0$ is the dissolution rate constant and $\psi(c_{12})$ is defined by

$$\psi(c_{12}) = \begin{cases} \{0\} & \text{if } c_{12} < 0, \\ [0,1] & \text{if } c_{12} = 0, \\ \{1\} & \text{if } c_{12} > 0. \end{cases} \tag{2.3.3}$$

Therefore the equation for immobile species is

$$\frac{\partial c_{12}}{\partial t} \in (R_p - k_d \psi(c_{12})). \tag{2.3.4}$$

In this work, we consider only on dissolution together with diffusion and reaction of chemical species and from here on precipitation is no longer considered. However, interested readers can look into the works of Knabner and van Duijn (cf. [Kna91], [Kna86], [KvDH95], [KvD96] etc.) and references therein for modeling and mathematical analysis of models involving precipitation and dissolution.

2.4 Diffusion-Reaction Models

Let $\Omega \subset \mathbb{R}^n$ be the given porous medium. Assume that Ω is a bounded domain. Let Ω^p and Ω^s denote the pore space and union of the solid parts such that $\Omega = \Omega^p \cup \Omega^s$ and $\bar{\Omega}^p \cap \Omega^s = \phi$, see figure 1.1.1. Suppose $\partial\Omega$ and Γ^* denote the boundary of the domain Ω and union of the boundaries of the solid parts respectively. We define $\partial\Omega^p := \partial\Omega \cup \Gamma^*$. Both $\partial\Omega$ and Γ^* are assumed to be sufficiently smooth. For a $T > 0$, $[0,T)$ denotes the time interval.

2.4.1 Model M1

Let I number of mobile species be present in the pore space Ω^p (see figure 2.4.1). These species diffuse and react with each other. All these reactions are reversible. We assume that the fluid velocity is 0, i.e., there is no advection. The reaction is shown below

$$\tau_{1j}X_1 + \tau_{2j}X_2 + \ldots + \tau_{Ij}X_I \rightleftharpoons \nu_{1j}X_1 + \nu_{2j}X_2 + \ldots + \nu_{Ij}X_I, \text{ for } 1 \le j \le J, \tag{2.4.1}$$

where X_i, for $1 \le i \le I$, denotes the chemical species involved in J number of reactions. The stoichiometric coefficients $-\tau_{ij} \in \mathbb{Z}_0^-$ and $\nu_{ij} \in \mathbb{Z}_0^+$ respectively. Let u_i denote the concentration of X_i for $1 \le i \le I$. Then the system of diffusion-reaction equations of these species is given as[3]

$$\frac{\partial u}{\partial t} - \nabla \cdot \bar{D}\nabla u = SR(u) \qquad \text{in} \quad (0,T) \times \Omega^p, \tag{2.4.2}$$

[3]The reaction rates (given by mass action law) are of the form (2.4.7) which is motivated from the work of Kräutle in [Krä08].

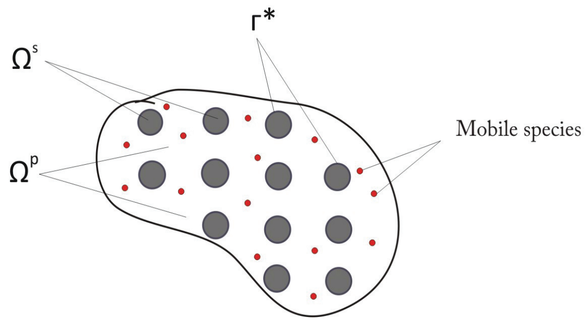


Figure 2.4.1: Model M1 with mobile species in Ω^p.

where $u = (u_1, u_2, ..., u_I)$, and $SR(u)$ is the reaction term. Here $\bar{D} := diag(d_1, d_2, ..., d_I)$ is the diagonal positive definite matrix of diffusion coefficients d_i for $1 \le i \le I$ and S is the $I \times J$-*th* order stoichiometric matrix with entries $s_{ij} = \nu_{ij} - \tau_{ij}$, i.e.,

$$S = (s_{ij})_{\substack{1\le i\le I\\ 1\le j\le J}} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1J} \\ s_{21} & s_{22} & \dots & s_{1J} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ s_{I1} & s_{I2} & \dots & s_{IJ} \end{bmatrix}_{I\times J}, \tag{2.4.3}$$

and $R = (R_j)_{1\le j\le J}$ is the J-*th* order reaction rate vector which is given as

$$R_j(u) = R_j^f(u) - R_j^b(u), \tag{2.4.4}$$

where

$$R_j^f(u) = \text{forward reaction rate} = k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} \tag{2.4.5}$$

and

$$R_j^b(u) = \text{backward reaction rate} = k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}, \tag{2.4.6}$$

where $k_j^f,\ k_j^b > 0$ are forward and backward reaction rate factors respectively. Therefore the reaction rate term for the i-*th* species is given by

$$\begin{aligned} (SR(u))_i &= \sum_{j=1}^{J} s_{ij} R_j(u) \\ &= \sum_{j=1}^{J} s_{ij} \left(R_j^f(u) - R_j^b(u) \right) \\ &= \sum_{j=1}^{J} s_{ij} \left(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \right). \end{aligned} \tag{2.4.7}$$

We suppose that the species present in the fluid have no interaction with the boundaries $\partial\Omega$ and Γ^*, in other words, no flux is entering or leaving the domain Ω through $\partial\Omega$ and Γ^*.

This can be mathematically written as

$$-\bar{D}\,\nabla u\cdot\vec{n} \;=\; 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{2.4.8}$$
$$-\bar{D}\,\nabla u\cdot\vec{n} \;=\; 0 \quad \text{on } (0,T)\times\Gamma^{*}. \tag{2.4.9}$$

The BCs (2.4.8) and (2.4.9) can be rewritten as

$$-\bar{D}\,\nabla u\cdot\vec{n} \;=\; 0 \quad \text{on } (0,T)\times\partial\Omega^{p}. \tag{2.4.10}$$

Initially for $t=0$, we assume $u(0,x) \;=\; u_0(x)$ in Ω^p, where $u_0(x)>0$ componentwise, i.e.,

$$u_{0_i}(x) \;>\; 0 \text{ for all } i \;=\; 1,2,\ldots,I. \tag{2.4.11}$$

For technical reasons, we replace the matrix $\bar{D}$ by a strictly positive constant D and from here on we assume

$$\boxed{\bar{D}:=D>0.} \tag{2.4.12}$$

Therefore the diffusion-reaction model is given by

$$\begin{aligned}
\frac{\partial u}{\partial t}-\nabla\cdot D\nabla u &= SR(u) && \text{in } (0,T)\times\Omega^{p}, && (2.4.13)\\
-D\,\nabla u\cdot\vec{n} &= 0 && \text{on } (0,T)\times\Gamma^{*}, && (2.4.14)\\
-D\,\nabla u\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega, && (2.4.15)\\
u(0,x) &= u_0(x) && \text{in } \Omega^{p}. && (2.4.16)
\end{aligned}$$

We are mainly interested in the global solution of the problem (2.4.13)-(2.4.16), which is shown in chapter 4. In order to prove the existence of the global solution of this problem, we need the assumption (2.4.12) and this is the price which we have to pay at here. There are existence results for the global solution of a system of diffusion-reaction equations for some special situations (see [Pie10], [PS97]), but to our knowledge the existence of the global solution for (2.4.13)-(2.4.16) with $I(>2)$ different diffusion coefficients is still unknown.

2.4.2 Model M2

Let Ω, Γ^*, $\partial\Omega$ and $\partial\Omega^p$ be as in section 2.4.1. We incorporate the dissolution process, defined in section 2.3, in the previous model. Let $\vec{q}$ be the given velocity field of the fluid which is present in the pore space of the porous medium Ω such that

$$\nabla\cdot\vec{q} \;=\; 0 \quad \text{in } \Omega^{p}, \tag{2.4.17}$$
$$\vec{q} \;=\; 0 \quad \text{on } \Gamma^{*}. \tag{2.4.18}$$

Let I_1 number of mobile species present in the fluid. We refer to these I_1 species as *type I* species. Let I_2 number of immobile species (crystals) present on the surface of the solid parts. Due to the presence of the fluid in Ω^p, immobile species interact with the fluid on Γ^*, i.e., the dissolution of immobile species takes place on the surface of the solid parts. Suppose that a number of I_2 mobile species is supplied by immobile species via dissolution. We call these I_2 mobile species as *type II* species. Confer the figure 2.4.2. Both *type I* and *type II* species transport inside the domain by the effect of diffusion and advection and they react with each other under the following reaction:

$$\left.\begin{array}{c}
\tau_{1j}X_1+\tau_{2j}X_2+\ldots+\tau_{I_1j}X_{I_1}+\kappa_{1j}Y_1+\kappa_{2j}Y_2+\ldots+\kappa_{I_2j}Y_{I_2}\\
\downharpoonleft\upharpoonright\\
\bar{\tau}_{1j}X_1+\bar{\tau}_{2j}X_2+\ldots+\bar{\tau}_{I_1j}X_{I_1}+\bar{\kappa}_{1j}Y_1+\bar{\kappa}_{2j}Y_2+\ldots+\bar{\kappa}_{I_2j}Y_{I_2},
\end{array}\right\} \tag{2.4.19}$$

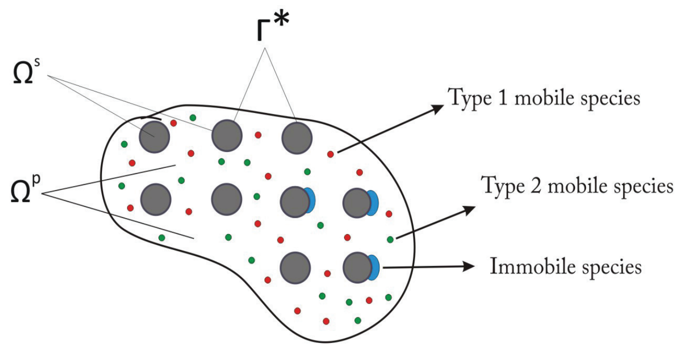


Figure 2.4.2: Model M2 with mobile species in Ω^p and immobile species on Γ^*.

where $1 \leq j \leq J$. For all $i = 1,2,...,I_1$, $k = 1,2,...,I_2$, X_i and Y_k denote *type I* and *type II* species respectively. The stoichiometric coefficients $-\tau_{ij}$, $-\kappa_{ij} \in \mathbb{Z}_0^-$ and $\bar{\tau}_{ij}$, $\bar{\kappa}_{ij} \in \mathbb{Z}_0^+$ respectively. We define two stoichiometric matrix S_1 and S_2 of order $I_1 \times J$*-th* and $I_2 \times J$*-th* whose entries are $s_{ij} = \bar{\tau}_{ij} - \tau_{ij}$ and $\nu_{ij} = \bar{\kappa}_{ij} - \kappa_{ij}$ respectively, i.e.,

$$S_1 = (s_{ij})_{\substack{1\leq i\leq I_1\\ 1\leq j\leq J}} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1J} \\ s_{21} & s_{22} & \dots & s_{1J} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ s_{I_1 1} & s_{I_1 2} & \dots & s_{I_1 J} \end{bmatrix}_{I_1 \times J} \tag{2.4.20}$$

and

$$S_2 = (\nu_{ij})_{\substack{1\leq k\leq I_2\\ 1\leq j\leq J}} = \begin{bmatrix} \nu_{11} & \nu_{12} & \dots & \nu_{1J} \\ \nu_{21} & \nu_{22} & \dots & \nu_{1J} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \nu_{I_2 1} & \nu_{I_2 2} & \dots & \nu_{I_2 J} \end{bmatrix}_{I_2 \times J} . \tag{2.4.21}$$

For $i = 1,2,...,I_1$, $k = 1,2,...,I_2$ and $m = 1,2,...,I_2$, let u_i, v_k and w_m denote the concentrations of *type I*, *type II* and immobile species. Then the systems of diffusion-reaction equations for *type I* and *type II* species are given as

$$\frac{\partial u}{\partial t} - \nabla\cdot(D_1\nabla u - \vec{q}u) = S_1 R(u,v) \qquad \text{in} \quad (0,T)\times\Omega^p \tag{2.4.22}$$

and

$$\frac{\partial v}{\partial t} - \nabla\cdot(D_2\nabla v - \vec{q}v) = S_2 R(u,v) \qquad \text{in} \quad (0,T)\times\Omega^p. \tag{2.4.23}$$

where D_1 and D_2 are diagonal positive definite matrices. The dissolution equation for immobile species is given as

$$\frac{\partial w}{\partial t} = -k_d z \qquad \text{on} \quad (0,T)\times\Gamma^*, \tag{2.4.24}$$

$$z \in \psi(w) \qquad \text{on} \quad (0,T)\times\Gamma^*, \tag{2.4.25}$$

where

$$\boxed{\psi(w_m) = \begin{cases} \{0\} & \text{if } w_m < 0, \\ [0,1] & \text{if } w_m = 0, \\ \{1\} & \text{if } w_m > 0. \end{cases}} \tag{2.4.26}$$

For reasons of mathematical necessity, like (2.4.12), we replace the matrices D_1 and D_2 by a strictly positive constant D and from here on we assume $D := D_1 := D_2 > 0$. Let $\partial\Omega := \partial\Omega_{in} \cup \partial\Omega_{out}$, where on $\partial\Omega_{in}$ and $\partial\Omega_{out}$ we prescribe the inflow and outflow boundary conditions for the *type I* and *type II* species. Since *type II* species are supplied by the dissolution process on Γ^*, the flux for the *type II* species on Γ^* is equal to the rate of change of immobile species on Γ^*, i.e., for the *type II* species, we have an additional boundary condition. The complete diffusion-reaction-dissolution model is given as[4]

For *type I* species:

$$\boxed{\begin{aligned}
\frac{\partial u}{\partial t} - \nabla\cdot(D\nabla u - \vec{q}u) &= S_1 R(u,v) && \text{in } (0,T)\times\Omega^p, \\
-(D\nabla u - \vec{q}u)\cdot\vec{n} &= d && \text{on } (0,T)\times\partial\Omega_{in}, \\
-D\nabla u\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{out}, \\
-D\nabla u\cdot\vec{n} &= 0 && \text{on } (0,T)\times\Gamma^*, \\
u(0,x) &= u_0(x), && \text{in } \Omega^p.
\end{aligned}}
\begin{aligned}
&(2.4.27)\\ &(2.4.28)\\ &(2.4.29)\\ &(2.4.30)\\ &(2.4.31)
\end{aligned}$$

where $d \le 0$ componentwise, i.e., $d_i \le 0$ for all $1 \le i \le I_1$.
For *type II* species:

$$\boxed{\begin{aligned}
\frac{\partial v}{\partial t} - \nabla\cdot(D\nabla v - \vec{q}v) &= S_2 R(u,v) && \text{in } (0,T)\times\Omega^p, \\
-(D\nabla v - \vec{q}v)\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{in}, \\
-D\nabla v\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{out}, \\
-D\nabla v\cdot\vec{n} &= \frac{\partial w}{\partial t} && \text{on } (0,T)\times\Gamma^*, \\
v(0,x) &= v_0(x) && \text{in } \Omega^p.
\end{aligned}}
\begin{aligned}
&(2.4.32)\\ &(2.4.33)\\ &(2.4.34)\\ &(2.4.35)\\ &(2.4.36)
\end{aligned}$$

For immobile species:

$$\boxed{\begin{aligned}
\frac{\partial w}{\partial t} &= -k_d z && \text{on } (0,T)\times\Gamma^*, \\
z &\in \psi(w) && \text{on } (0,T)\times\Gamma^*, \\
w(0,x) &= w_0(x) && \text{on } \Gamma^*,
\end{aligned}}
\begin{aligned}
&(2.4.37)\\ &(2.4.38)\\ &(2.4.39)
\end{aligned}$$

where $\psi(w)$ is given by (2.4.26) and the initial conditions are strictly positive, i.e., $u_0(x)$, $v_0(x)$ and $w_0(x) > 0$ componentwise. For the velocity $\vec{q}$, we assume the following conditions:

$$\nabla\cdot\vec{q} = 0 \text{ in } \Omega^p,\ -\vec{q}\cdot\vec{n} > 0 \text{ on } \partial\Omega_{in},\ -\vec{q}\cdot\vec{n} \le 0 \text{ on } \partial\Omega_{out} \text{ and } \vec{q} = 0 \text{ on } \Gamma^*. \quad (2.4.40)$$

The reaction rate term for the *i-th* species of *type I* is given by

$$\begin{aligned}
(S_1 R_j(u,v))_i &= \sum_{j=1}^{J} s_{ij}\left(R_j^f(u,v) - R_j^b(u,v)\right) \\
&= \sum_{j=1}^{J} s_{ij}\left(k_j^f \prod_{\substack{r=1\\ s_{rj}<0}}^{I_1} u_r^{-s_{rj}} \prod_{\substack{l=1\\ \nu_{lj}<0}}^{I_2} v_l^{-\nu_{lj}} - k_j^b \prod_{\substack{r=1\\ s_{rj}>0}}^{I_1} u_r^{s_{rj}} \prod_{\substack{l=1\\ \nu_{lj}>0}}^{I_2} v_l^{\nu_{lj}}\right). \qquad (2.4.41)
\end{aligned}$$

[4] The proposed mathematical model is motivated from the works in [vDP04], [KvDH95], [Krä08].

Similarly, the reaction rate term for the *k-th* species of *type II* is given as

$$
\begin{aligned}
(S_2 R_j(u,v))_k &= \sum_{j=1}^{J} \nu_{kj} \left(R_j^f(u,v) - R_j^b(u,v) \right) \\
&= \sum_{j=1}^{J} \nu_{kj} \left(k_j^f \prod_{\substack{r=1 \\ s_{rj}<0}}^{I_1} u_r^{-s_{rj}} \prod_{\substack{l=1 \\ \nu_{lj}<0}}^{I_2} v_l^{-\nu_{lj}} - k_j^b \prod_{\substack{r=1 \\ s_{rj}>0}}^{I_1} u_r^{s_{rj}} \prod_{\substack{l=1 \\ \nu_{lj}>0}}^{I_2} v_l^{\nu_{lj}} \right),
\end{aligned}
\tag{2.4.42}
$$

where k_j^f and $k_j^b > 0$ are the forward and backward reaction rate factors. In next section, we derive the models M1 and M2 at the microscopic scale.

2.5 Scaling

2.5.1 The ε-periodic Approximation of Ω

We begin this section by making some assumptions on our porous medium Ω introduced in section 2.4. Let $Y = (0,1)^n \subset \mathbb{R}^n$ be the unit representative cell which is composed of a solid part Y^s and a pore part Y^p such that $Y = Y^s \cup Y^p$ and $\bar{Y}^s \subset Y$ (see figure 2.5.1). Let Γ be the sufficiently smooth boundary of Y^s.

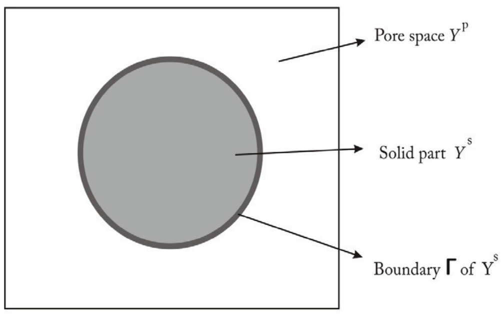


Figure 2.5.1: An example of the representative cell Y.

Let $\chi(y)$ be the Y-periodic characteristic (indicator) function of Y^p defined by

$$
\begin{aligned}
\chi(y) &= 1 \ \text{ for } y \in Y^p, \\
&= 0 \ \text{ for } y \in Y - Y^p.
\end{aligned}
$$

The domain Ω is assumed to be periodic and is covered by a finite union of the cells Y. In order to avoid the technical difficulties, we postulate that:

- solid parts do not touch the boundary $\partial\Omega$,
- solid parts do not touch each other,
- solid parts do not touch the boundary of Y.

For $n = 2$, the disconnectedness of solid parts does not disrupt the generality as the connection of two solid parts will imply the blocking of porous samples, see figure 2.5.2. However, for $n \geq 3$, the disconnectedness of the solid parts is actually an assumption, since the connection between the two solid parts is possible without violating the periodicity of the domain, see figure 2.5.3.

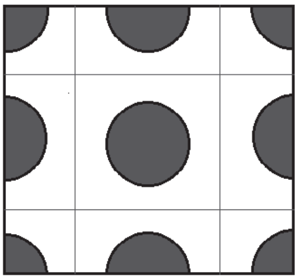


Figure 2.5.2: Disconnected solid parts in 2D.

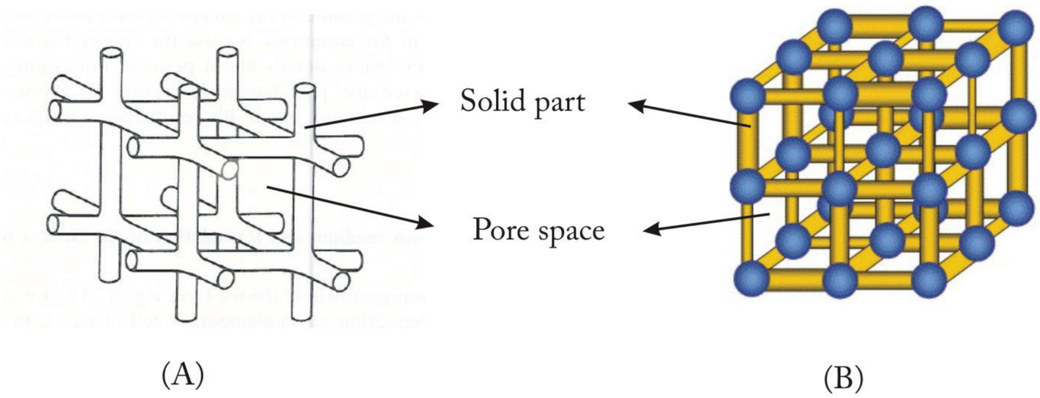


Figure 2.5.3: Connected solid parts in 3D.[5]

For any $m = (m_1, m_2, ..., m_n) \in \mathbb{Z}^n$, we define

$$Y_m := Y + \sum_{l=1}^{n} m_l e_l, \tag{2.5.1}$$

$$Y_m^p := Y^p + \sum_{l=1}^{n} m_l e_l, \tag{2.5.2}$$

$$Y_m^s := Y^s + \sum_{l=1}^{n} m_l e_l, \tag{2.5.3}$$

$$\Gamma_m := \Gamma + \sum_{l=1}^{n} m_l e_l, \tag{2.5.4}$$

where e_l is the *l-th* unit vector, such that

$$\Omega \subset \cup_{m\in\mathbb{Z}^n} Y_m, \tag{2.5.5}$$

$$\Omega^p \subset \cup_{m\in\mathbb{Z}^n} Y_m^p, \tag{2.5.6}$$

$$\Omega^s \subset \cup_{m\in\mathbb{Z}^n} Y_m^s, \tag{2.5.7}$$

$$\Gamma^* \subset \cup_{m\in\mathbb{Z}^n} \Gamma_m. \tag{2.5.8}$$

Here we follow the notations introduced in [Mil92]. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero. If there is no confusion, we drop the suffix 'n'. Let Ω is covered by a finite union of εY_m cells such that $\varepsilon Y_m \subset \Omega$, where $m \in \mathbb{Z}^n$. To be more

[5] The figures A and B are taken from Asymptotic Analysis, Vol. 2, pp 203-222, 1989 and Advances in Chemical Engineering, Vol. 30, pp 137-203, 2005 respectively.

precise, it is assumed that there is an $\varepsilon_0 > 0$, called the natural scaling parameter, such that Ω is covered by the finite union of $\varepsilon_0 Y_m$ cells. However, for the homogenization, we consider the sequence of positive real numbers, ε to converge to 0 (see fig 2.5.4).

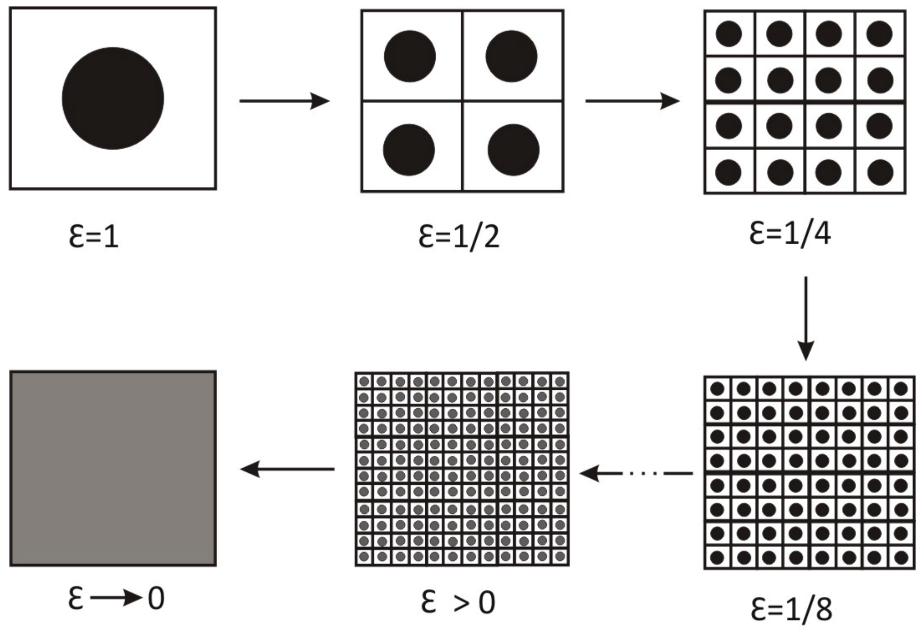


Figure 2.5.4: A schematic representation of periodic homogenization.

We further define

$$\begin{aligned}
\Omega_\varepsilon^p &:= \cup_{m\in\mathbb{Z}^n}\left\{\varepsilon Y_m^p : \varepsilon Y_m^p \subset \Omega\right\}, && (2.5.9)\\
\Omega_\varepsilon^s &:= \cup_{m\in\mathbb{Z}^n}\left\{\varepsilon Y_m^s : \varepsilon Y_m^s \subset \Omega\right\}, && (2.5.10)\\
\Gamma_\varepsilon &:= \cup_{m\in\mathbb{Z}^n}\left\{\varepsilon \Gamma_m : \varepsilon \Gamma_m \subset \Omega\right\}, && (2.5.11)\\
\partial\Omega_\varepsilon^p &:= \partial\Omega \cup \Gamma_\varepsilon, && (2.5.12)
\end{aligned}$$

see figure 2.5.5.

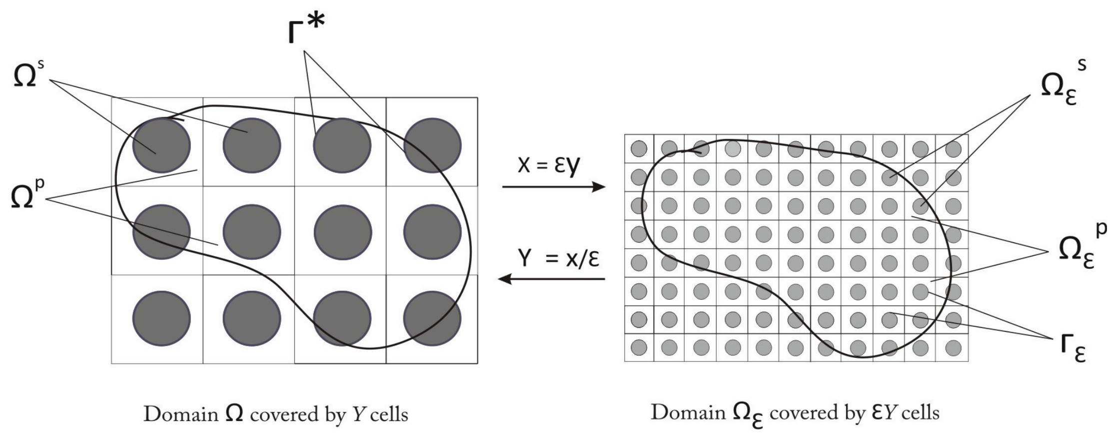


Figure 2.5.5: ε-periodic scaling of the domain Ω.

We denote dx and dy as the volume elements in Ω and in Y, and $d\sigma_y$ and $d\sigma_x$ as the surface elements on Γ and on Γ_ε respectively. Due to Y-periodicity, the characteristic function of the domain Ω_ε^p in the domain Ω is given by

$$\chi^\varepsilon(x) \;=\; \chi(\frac{x}{\varepsilon}) \tag{2.5.13}$$

and is defined as

$$\begin{aligned}\chi^\varepsilon(x) &= 1 \text{ for } x \in \Omega_\varepsilon^p,\\ &= 0 \text{ for } x \in \Omega - \Omega_\varepsilon^p.\end{aligned} \tag{2.5.14}$$

2.5.2 Setting of Model M1 at the Micro Scale

Nondimensionalization: The description of model MI at the microscopic scale using the scaling parameter ε can be motivated from the nondimensionalization of the equations (2.4.13)-(2.4.16). Assume that u_{ref} is the reference concentration of the mobile species which can be an upper bound of the concentration and may be given from physical considerations or maximum estimates. Let l_{ref} be the reference microscopic length (e.g., a typical pore diameter) and L_{ref} denote the reference macroscopic length (e.g., the diameter of the domain Ω). Also assume that T_{ref} $(= \frac{L_{ref}^2}{D})$ is the reference time. We set

$$\begin{aligned} u_\varepsilon &= \frac{u}{u_{ref}}, & \bar{x} &= \frac{x}{L_{ref}}, & \bar{t} &= \frac{t}{T_{ref}},\\ \bar{D} &= \frac{DT_{ref}}{L_{ref}^2}, & \varepsilon &= \frac{l_{ref}}{L_{ref}}. \end{aligned} \tag{2.5.15}$$

We denote the scaled domain Ω^p and interface Γ^* by Ω_ε^p and Γ_ε respectively. We use the old notation D for $\bar{D}$, i.e., $D = \bar{D}$. A straightforward simplification will yield the required microscopic description of (2.4.13)-(2.4.16) which is given by

$$\boxed{\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \nabla\cdot D\nabla u_\varepsilon &= SR(u_\varepsilon) && \text{in} \quad (0,T)\times\Omega_\varepsilon^p, && (2.5.16)\\ u_\varepsilon(0,x) &= u_0(x) && \text{in} \quad \Omega_\varepsilon^p, && (2.5.17)\\ -D\nabla u_\varepsilon\cdot\vec{n} &= 0 && \text{on} \quad (0,T)\times\partial\Omega, && (2.5.18)\\ -D\nabla u_\varepsilon\cdot\vec{n} &= 0 && \text{on} \quad (0,T)\times\Gamma_\varepsilon. && (2.5.19) \end{aligned}}$$

We denote this problem by (P_ε^1). In chapter 3, we give the notion of weak solution of (P_ε^1) in some appropriate sense and we prove the existence of a unique positive global weak solution of this problem in chapter 4. The homogenization of (P_ε^1) is also shown in chapter 4.

2.5.3 Setting of Model M2 at the Micro Scale

Nondimensionalization: In this section, we give the microscopic description of model M2. For this model, we adopt the nondimensionalization technique from [vDP04]. We nondimensionalize the equations (2.4.27)-(2.4.39) in the following way: Let u_{ref}, v_{ref} and w_{ref} be the characteristic concentrations of *type I*, *type II* and immobile species respectively which can be the upper bounds of the concentrations. Further assume that q_{ref}, L_{ref} and $T_{ref}(=\frac{L_{ref}}{q_{ref}})$ are the characteristic velocity, length and time respectively. We set

$$\begin{array}{llllll} u_\varepsilon = \frac{u}{u_{ref}}, & v_\varepsilon = \frac{v}{v_{ref}}, & w_\varepsilon = \frac{w}{w_{ref}}, \\ \vec{q}_\varepsilon = \frac{\vec{q}}{q_{ref}}, & \bar{D} = \frac{D}{L_{ref}q_{ref}}, & \bar{k}_d = \frac{k_d L_{ref}}{q_{ref}w_{ref}}, \\ \bar{x} = \frac{x}{L_{ref}}, & \bar{t} = \frac{t}{T_{ref}}, & \varepsilon = \frac{w_{ref}}{L_{ref}u_{ref}}, \\ \bar{d} = \frac{d}{q_{ref}u_{ref}}. \end{array} \tag{2.5.20}$$

We denote the scaled domain Ω^p and interface Γ^* by Ω_ε^p and Γ_ε respectively. We use the old notations D, k_d, and d for $\bar{D}$, $\bar{k}_d$ and $\bar{d}$ respectively. With the help of (2.5.20), the required microscopic description of (2.4.27)-(2.4.39) is given by

Equations for *type I* species :

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla\cdot(D\nabla u_\varepsilon - \vec{q}_\varepsilon u_\varepsilon) = S_1 R(u_\varepsilon, v_\varepsilon) \quad \text{in } (0,T)\times\Omega_\varepsilon^p, \tag{2.5.21}$$

$$-(D\nabla u_\varepsilon - \vec{q}_\varepsilon u_\varepsilon)\cdot\vec{n} = d \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{2.5.22}$$

$$-D\nabla u_\varepsilon\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{2.5.23}$$

$$-D\nabla u_\varepsilon\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.5.24}$$

$$u_\varepsilon(0,x) = u_0(x), \quad \text{in } \Omega_\varepsilon^p, \tag{2.5.25}$$

$$\text{where } d_i \le 0 \text{ for all } 1 \le i \le I_1. \tag{2.5.26}$$

Equations for *type II* species :

$$\frac{\partial v_\varepsilon}{\partial t} - \nabla\cdot(D\nabla v_\varepsilon - \vec{q}_\varepsilon v_\varepsilon) = S_2 R(u_\varepsilon, v_\varepsilon) \quad \text{in } (0,T)\times\Omega_\varepsilon^p, \tag{2.5.27}$$

$$-(D\nabla v_\varepsilon - \vec{q}_\varepsilon v_\varepsilon)\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{2.5.28}$$

$$-D\nabla v_\varepsilon\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{2.5.29}$$

$$-D\nabla v_\varepsilon\cdot\vec{n} = \varepsilon\frac{\partial w_\varepsilon}{\partial t} \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.5.30}$$

$$v_\varepsilon(0,x) = v_0(x), \quad \text{in } \Omega_\varepsilon^p. \tag{2.5.31}$$

Equations for immobile species:

$$\frac{\partial w_\varepsilon}{\partial t} = -k_d z \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.5.32}$$

$$z \in \psi(w_\varepsilon) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.5.33}$$

$$w_\varepsilon(0,x) = w_0(x) \quad \text{on } \Gamma_\varepsilon, \tag{2.5.34}$$

where

$$\psi(w_{\varepsilon_m}) = \begin{cases} \{0\} & \text{if } w_{\varepsilon_m} < 0, \\ [0,1] & \text{if } w_{\varepsilon_m} = 0, \\ \{1\} & \text{if } w_{\varepsilon_m} > 0. \end{cases} \tag{2.5.35}$$

The velocity $\vec{q}_\varepsilon$ satisfies:

$$\nabla\cdot\vec{q}_\varepsilon = 0 \text{ in } \Omega_\varepsilon^p,\ -\vec{q}_\varepsilon\cdot\vec{n} > 0 \text{ on } \partial\Omega_{in},\ -\vec{q}_\varepsilon\cdot\vec{n} \le 0 \text{ on } \partial\Omega_{out} \text{ and } \vec{q}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon. \tag{2.5.36}$$

We denote the problem (2.5.21)-(2.5.35) by (P_ε^2). The notion of weak solution for (P_ε^2) is given in chapter 3. The existence of a unique positive global weak solution of (P_ε^2) and its homogenization are shown in chapter 4.

Chapter 3

Mathematical Preliminaries

In this chapter, we collect some mathematical tools which are required to analyze the problems (P_ε^1) and (P_ε^2) in the next chapter. In section 3.1, we introduce the function spaces such as L^p*-spaces*, Sobolev spaces and their duals. In section 3.2, we give the notion of weak formulations for (P_ε^1) and (P_ε^2), respectively. We present a very short overview of *maximal parabolic regularity* of elliptic operators in section 3.3. Some extension and embedding theorems for the domain Ω_ε^p are proved in section 3.4. In sections 3.5 and 3.6, we present a short overview of *two-scale convergence* and *periodic unfolding* respectively.

3.1 Function Spaces

3.1.1 Function Spaces on Ω

Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. As usual, $L^p(\Omega)$ is the set of all equivalence classes of real-valued functions $u(.)$ such that $u(x)$ is defined for almost every $x \in \Omega$, is measurable and $|u(\cdot)|^p$ is Lebesgue integrable. $L^p(\Omega)$ is a Banach space w.r.t. the norm

$$||u||_{L^p(\Omega)} = \begin{cases} \left[\int_\Omega |u(x)|^p \, dx\right]^{\frac{1}{p}} & \text{for} \quad 1 \leq p < \infty \quad , \\ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| & \text{for} \quad p = \infty \quad . \end{cases} \tag{3.1.1}$$

The space $H^{1,p}(\Omega)$ is the usual Sobolev space w.r.t. the norm

$$||u||_{H^{1,p}(\Omega)} = \begin{cases} \left[||u||^p_{L^p(\Omega)} + ||\nabla u||^p_{L^p(\Omega)}\right]^{\frac{1}{p}} & \text{for} \quad 1 \leq p < \infty \quad , \\ \operatorname*{ess\,sup}_{x\in\Omega} [|u(x)| + |\nabla u(x)|] & \text{for} \quad p = \infty \quad . \end{cases} \tag{3.1.2}$$

The duality pairing between $H^{1,q}(\Omega)$ and $H^{1,q}(\Omega)^*$ is denoted by $\langle . \, , . \rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)}$. We define the continuous embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$ as

$$\langle f, v\rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} = \langle f, v\rangle_{L^p(\Omega)\times L^q(\Omega)} \text{ for } f \in L^p(\Omega),\, v \in H^{1,q}(\Omega). \tag{3.1.3}$$

For $k \in \mathbb{Z}_0^+$, the space $C^k(\bar{\Omega})$ denotes the Banach space of all k-times continuously differentiable functions w.r.t. the norm

$$||u||_{C^k(\bar{\Omega})} = \sum_{|\alpha|\leq k} \sup_{x\in\bar{\Omega}} |D^\alpha u(x)| \,. \tag{3.1.4}$$

Suppose that $0 < \gamma \leq 1$. The space $C^\gamma(\bar{\Omega})$ consists of all functions $u \in C(\bar{\Omega})$ such that

$$||u||_{C^\gamma(\bar{\Omega})} = ||u||_{C(\bar{\Omega})} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\} < \infty. \tag{3.1.5}$$

The space $C^{\gamma}(\bar{\Omega})$ is called the Hölder space. We introduce the *Sobolev-Bochner space* as

$$\begin{aligned} F \;:=\; F^p(\Omega) \;&:=\; \left\{u \in L^p((0,T);H^{1,p}(\Omega)) : \frac{du}{dt} \in L^p((0,T);H^{1,q}(\Omega)^*)\right\} \\ &=\; H^{1,p}((0,T);H^{1,q}(\Omega)^*) \cap L^p((0,T);H^{1,p}(\Omega)). \end{aligned} \tag{3.1.6}$$

and for any $u \in F$,

$$||u||_F \;=\; ||u||_{L^p((0,T);H^{1,p}(\Omega))} + ||u||_{L^p((0,T);H^{1,q}(\Omega)^*)} + \left|\left|\frac{du}{dt}\right|\right|_{L^p((0,T);H^{1,q}(\Omega)^*)}, \tag{3.1.7}$$

where $\frac{du}{dt}$ is the distributional time derivative of u. For $0<\theta<1$, let

$$\Big(H^{1,q}(\Omega)^*, H^{1,p}(\Omega)\Big)_{\theta,p} \;-\; \text{the real-interpolation space between } H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega), \tag{3.1.8}$$

$$\Big[H^{1,q}(\Omega)^*, H^{1,p}(\Omega)\Big]_{\theta} \;-\; \text{the complex-interpolation space between } H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega) \tag{3.1.9}$$

endowed with one of their usual norms (cf. [BL76], [Tri95], [Lun95], [Has06]).

Theorem 3.1.1. *The space* $F \hookrightarrow C([0,T];(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p})$.

Proof. See theorem 4.10.2 in [Ama95]. See also proposition 1.2.10 and remark 1.2.11 in [Lun95]. ♦

Theorem 3.1.2. *Let* $p > n+2$, *then* $F \hookrightarrow\hookrightarrow L^{\infty}((0,T)\times\Omega)$.

Proof. Step 1.: We notice that

$$\begin{aligned} ||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*} \;&=\; \left|\left|\int_{t_0}^{t} v^{'}(s)\,ds\right|\right|_{H^{1,q}(\Omega)^*} \\ &\leq\; \int_{t_0}^{t} \left|\left|v^{'}(s)\right|\right|_{H^{1,q}(\Omega)^*} ds \\ &\leq\; \left[\int_{t_0}^{t} \left|\left|v^{'}(s)\right|\right|^p_{H^{1,q}(\Omega)^*} ds\right]^{\frac{1}{p}} \left[\int_{t_0}^{t} ds\right]^{\frac{1}{q}} \\ &\leq\; ||v||_{H^{1,p}((0,T);H^{1,q}(\Omega)^*)} \, |t-t_0|^{\frac{1}{q}} \\ \Longrightarrow \quad \frac{||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\frac{1}{q}}} \;&\leq\; ||v||_{H^{1,p}((0,T);H^{1,q}(\Omega)^*)}. \end{aligned} \tag{3.1.10}$$

This implies $H^{1,p}((0,T);H^{1,q}(\Omega)^*) \hookrightarrow C^{\delta}([0,T];H^{1,q}(\Omega)^*)$, where $\delta = \frac{1}{q} = 1-\frac{1}{p}$.

Step 2.: The condition $p > n+2$ implies $\frac{1}{2}+\frac{n}{2p} < 1-\frac{1}{p}$. Choose $\lambda \in \left(\left(\frac{1}{2}+\frac{n}{2p}\right)\left(1-\frac{1}{p}\right)^{-1}, 1\right)$ and set $\eta := \lambda(1-\frac{1}{p})$. Then by reiteration theorem on real-interpolation

$$\begin{aligned} &\frac{||v(t)-v(t_0)||_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\eta,1}}}{|t-t_0|^{\delta(1-\lambda)}} \\ =\;& \frac{||v(t)-v(t_0)||_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\lambda(1-\frac{1}{p}),1}}}{|t-t_0|^{\delta(1-\lambda)}} \\ =\;& \frac{||v(t)-v(t_0)||_{(H^{1,q}(\Omega)^*,(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{(1-\frac{1}{p}),p})_{\lambda,1}}}{|t-t_0|^{\delta(1-\lambda)}} \\ \leq\;& C\,\frac{||v(t)-v(t_0)||^{1-\lambda}_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\delta(1-\lambda)}} \times ||v(t)-v(t_0)||^{\lambda}_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \end{aligned}$$

$$\leq C\left(\frac{||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\delta}}\right)^{1-\lambda} \times 2\sup_{t\in(0,T)} ||v(t)||^{\lambda}_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}}$$
$$\leq C\left(\frac{||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\delta}}\right)^{1-\lambda} \times ||v||^{\lambda}_{C([0,T];(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p})}. \tag{3.1.11}$$

Therefore by step 1 and theorem 3.1.1, it follows that $F \hookrightarrow C^{\beta}([0,T];(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\eta,1})$, where $\frac{1}{2}+\frac{n}{2p}<\eta<1-\frac{1}{p}$ and $\beta=\delta(1-\lambda)$.
Step 3.: We have the following embedding (cf. theorem 1.3.3.d in [Tri95] and corollary 5.28 in [KR13])

$$(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\eta,1} \hookrightarrow (H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\eta,p} \hookrightarrow H^{2\eta-1,p}(\Omega) \hookrightarrow C^{\alpha}(\bar{\Omega}),$$

where $\alpha=2\eta-1-\frac{n}{p}>0$. Therefore combining the steps 2 and 3, we obtain

$$\begin{aligned} F \hookrightarrow C^{\beta}([0,T];C^{\alpha}(\bar{\Omega})) &\hookrightarrow C^{\sigma}([0,T]\times\bar{\Omega}) \\ &\hookrightarrow\hookrightarrow L^{\infty}((0,T)\times\Omega), \text{ where } \sigma = \min(\alpha,\beta). \end{aligned}$$

♦

Theorem 3.1.3. *Let $p>n+2$. Then $(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p} \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$.*

Proof. Let us denote $E_0=H^{1,q}(\Omega)^*$, $E_1=H^{1,p}(\Omega)$ and $E_{1-\frac{1}{p},p}=(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}$. By lemma 3.4 in [GGKR00]: $[E_0,E_1]_{\frac{1}{2}} \hookrightarrow L^p(\Omega)$. From this and reiteration theorem on real-interpolation, we obtain

$$E_{1-\frac{1}{p},p} = ([E_0,E_1]_{\frac{1}{2}},[E_0,E_1]_1)_{1-\frac{2}{p},p} \hookrightarrow (L^p(\Omega),H^{1,p}(\Omega))_{1-\frac{2}{p},p} = H^{1-\frac{2}{p},p}(\Omega).$$

There exists a $t>0$ such that $p > n+2 \Rightarrow 1-\frac{n+2}{p} > t > 0 \Rightarrow 1-\frac{2}{p} > t+\frac{n}{p}$. From theorem 4.6.1 (e) in Triebel [Tri95]: $H^{1-\frac{2}{p},p}(\Omega) \hookrightarrow C^t(\bar{\Omega})$. Since $C^t(\bar{\Omega}) \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$, $H^{1-\frac{2}{p},p}(\Omega) \hookrightarrow C^t(\bar{\Omega}) \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$. Therefore $(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p} \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$. ♦

Now we introduce the norms on the vector-valued function spaces. Let $I\in\mathbb{N}$ and $u:\Omega\to\mathbb{R}^I$ be a vector-valued function. We define

$$[L^p(\Omega)]^I := \underbrace{L^p(\Omega)\times L^p(\Omega)\times\ldots\times L^p(\Omega)}_{I\text{-times}} \tag{3.1.12}$$

and for $u\in[L^p(\Omega)]^I$ the corresponding norm is given as

$$|||u|||_{[L^p(\Omega)]^I} := \left[\sum_{i=1}^{I}||u_i||^p_{L^p(\Omega)}\right]^{\frac{1}{p}}. \tag{3.1.13}$$

Similary,

$$|||u|||_{[L^{\infty}(\Omega)]^I} := \max_{1\leq i\leq I}||u_i||_{L^{\infty}(\Omega)}, \tag{3.1.14}$$

$$|||u|||_{[H^{1,p}(\Omega)]^I} := \left[\sum_{i=1}^{I}||u_i||^p_{H^{1,p}(\Omega)}\right]^{\frac{1}{p}}, \tag{3.1.15}$$

$$|||u|||_{[H^{1,\infty}(\Omega)]^I} := \max_{1\leq i\leq I}||u_i||_{H^{1,\infty}(\Omega)}, \tag{3.1.16}$$

$$|||u|||_{[H^{1,q}(\Omega)^*]^I} \;:=\; \left[\sum_{i=1}^{I} ||u_i||^p_{H^{1,q}(\Omega)^*}\right]^{\frac{1}{p}}, \tag{3.1.17}$$

$$|||u|||_{[(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}]^I} \;:=\; \left[\sum_{i=1}^{I} ||u||^p_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}}\right]^{\frac{1}{p}}. \tag{3.1.18}$$

For any two positive integers I_1 and I_2, we define

$$\mathcal{F}^u_p \;:=\; \left[H^{1,p}((0,T);H^{1,q}(\Omega)^*)\cap L^p((0,T);H^{1,p}(\Omega))\right]^{I_1} \tag{3.1.19}$$

such that for $u \in \mathcal{F}^u_p$,

$$|||u|||_{\mathcal{F}^u_p} \;:=\; \left[\sum_{i=1}^{I_1} ||u_i||^p_F\right]^{\frac{1}{p}}. \tag{3.1.20}$$

Similarly,

$$\mathcal{G}^v_p \;:=\; \left[H^{1,p}((0,T);H^{1,q}(\Omega)^*)\cap L^p((0,T);H^{1,p}(\Omega))\right]^{I_2}, \tag{3.1.21}$$

$$\mathcal{H}^w_p \;:=\; \left[H^{1,p}((0,T);L^p(\Gamma\times\Omega))\right]^{I_2}, \tag{3.1.22}$$

$$\mathcal{M}^z_\infty \;:=\; [L^\infty((0,T)\times\Gamma\times\Omega)]^{I_2}, \tag{3.1.23}$$

$$\mathcal{X}^u_p \;:=\; \left[(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}\right]^{I_1}, \tag{3.1.24}$$

$$\mathcal{X}^v_p \;:=\; \left[(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}\right]^{I_2}, \tag{3.1.25}$$

$$\mathcal{X}^w_\infty \;:=\; [L^\infty(\Gamma\times\Omega)]^{I_2}. \tag{3.1.26}$$

Furthermore, let V, H and V^* be the *Gelfand triple*, where V a Banach space, H a Hilbert space and V^* is the dual of V. Let H be identified with its own dual ($H \cong H^*$) and $V \overset{d}{\subset} H$, then $H \overset{d}{\subset} V^*$. Denote $\Xi := \left\{u \in L^p((0,T);V) : \frac{du}{dt} \in L^q((0,T);V^*)\right\}$. We have the following theorem:

Theorem 3.1.4. *Let V, H and V^* be as above. Then $\Xi \subset C([0,T];H)$ and the following rule of integration holds for any u, $v \in \Xi$ and any $0 \le t_1 \le t_2 \le T$:*

$$\int_{t_1}^{t_2} \frac{d}{dt}(u(t),v(t))_H\,dt \;=\; \int_{t_1}^{t_2}\left\langle \frac{du}{dt}, v(t)\right\rangle_{V^*\times V} dt + \int_{t_1}^{t_2}\left\langle u(t),\frac{dv}{dt}\right\rangle_{V\times V^*} dt.$$

Proof. Cf. lemma 7.3 in [Rou05]. ♦

3.1.2 Function Spaces on Ω^p_ε

The function spaces on the domain Ω^p_ε are defined in the analogous way as in section 3.1.1 by replacing the domain Ω by Ω^p_ε in the definitions of the function spaces. The spaces on Ω^p_ε are endowed with their usual norms as given in (3.1.1)-(3.1.9).

From section 2.5.1, we notice that the surface area of Γ_ε increases proportionally to $\frac{1}{\varepsilon}$, i.e., $|\Gamma_\varepsilon| \to \infty$ as $\varepsilon \to 0$. Keeping this in mind, the $L^p - L^q$ duality on Γ_ε is defined as

$$(u,v)_{L^p(\Gamma_\varepsilon)\times L^q(\Gamma_\varepsilon)} \;:=\; \varepsilon\int_{\Gamma_\varepsilon} u(x)v(x)\,d\sigma_x \text{ for } u \in L^p(\Gamma_\varepsilon) \text{ and } v \in L^q(\Gamma_\varepsilon), \tag{3.1.27}$$

and the space $L^p(\Gamma_\varepsilon)$ is furnished with the norm

$$||.||^p_{L^p(\Gamma_\varepsilon)} = \varepsilon\int_{\Gamma_\varepsilon} |.|^p\, d\sigma_x \quad \text{and} \quad ||.||_{L^\infty(\Gamma_\varepsilon)} = \operatorname*{ess\,sup}_{x\in\Gamma_\varepsilon} |.|. \tag{3.1.28}$$

The vector-valued functions and their respective norms on Ω^p_ε can be defined in the similar way as in (3.1.12)-(3.1.26). For the sake of simplicity, we use the following notations:

$$\mathcal{F}^u_\varepsilon := F^{I_1}_\varepsilon := \left[H^{1,p}((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)\cap L^p((0,T);H^{1,p}(\Omega^p_\varepsilon))\right]^{I_1}, \tag{3.1.29}$$

$$\mathcal{G}^v_\varepsilon := G^{I_2}_\varepsilon := \left[H^{1,p}((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)\cap L^p((0,T);H^{1,p}(\Omega^p_\varepsilon))\right]^{I_2}, \tag{3.1.30}$$

$$\mathcal{H}^w_\varepsilon := \left[H^{1,p}((0,T);L^p(\Gamma_\varepsilon))\right]^{I_2}, \tag{3.1.31}$$

$$\mathcal{M}^z_\varepsilon := [L^\infty((0,T);L^\infty(\Gamma_\varepsilon))]^{I_2}, \tag{3.1.32}$$

$$\mathcal{X}^u_{p_\varepsilon} := \left[(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}\right]^{I_1}, \tag{3.1.33}$$

$$\mathcal{X}^v_{p_\varepsilon} := \left[(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}\right]^{I_2}, \tag{3.1.34}$$

$$\mathcal{X}^w_{p_\varepsilon} := [L^\infty(\Gamma_\varepsilon)]^{I_2}. \tag{3.1.35}$$

3.2 Weak Formulation of (P^1_ε) and (P^2_ε)

We note that in case of (P^1_ε) $I_1 = I$, since there are only I mobile species present in Ω^p_ε.

Definition 3.2.1. *A function $u_\varepsilon \in \mathcal{F}^u_\varepsilon$ is said to be a weak solution of the problem (2.5.16)-(2.5.19) if it satisfies*

$$(i) \quad \left\langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} + \int_{\Omega^p_\varepsilon} \langle D\nabla u_\varepsilon(t,x), \nabla\phi(x)\rangle_I\, dx$$
$$= \langle SR(u_\varepsilon(t)), \phi\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I}$$
$$\text{for every } \phi\in[H^{1,q}(\Omega^p_\varepsilon)]^I \text{ and for a.e. } t. \tag{3.2.1}$$

$$(ii) \quad u_\varepsilon(0,x) = u_0(x). \tag{3.2.2}$$

Definition 3.2.2. *A quadruple $(u_\varepsilon, v_\varepsilon, w_\varepsilon, z) \in \mathcal{F}^u_\varepsilon\times\mathcal{G}^v_\varepsilon\times\mathcal{H}^w_\varepsilon\times\mathcal{M}^z_\varepsilon$ is said to be a weak solution of the problem (2.5.21)-(2.5.35) if it satisfies*

$$(i) \quad \left\langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^{I_1}\times[H^{1,q}(\Omega^p_\varepsilon)]^{I_1}} dt + \int_{\Omega^p_\varepsilon}\langle D\nabla_x u_\varepsilon(t,x), \nabla_x\phi(x)\rangle_{I_1}\, dx$$
$$+\int_{\partial\Omega_{in}}\langle (d-\vec{q}\cdot\vec{n})u_\varepsilon(t,x), \phi(x)\rangle_{I_1}\, ds + \int_{\Omega^p_\varepsilon}\langle \vec{q}\cdot\nabla u_\varepsilon(t,x), \phi(x)\rangle_{I_1}\, dx\, dt$$
$$= \langle S_1R(u_\varepsilon(t), v_\varepsilon(t)), \phi\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^{I_1}\times[H^{1,q}(\Omega^p_\varepsilon)]^{I_1}}, \tag{3.2.3}$$

$$(ii) \quad \left\langle \frac{\partial v_\varepsilon(t)}{\partial t}, \xi\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^{I_2}\times[H^{1,q}(\Omega^p_\varepsilon)]^{I_2}} + \int_{\Omega^p_\varepsilon}\langle D\nabla_x v_\varepsilon(t,x), \nabla_x\xi(x)\rangle_{I_2}\, dx$$
$$-\int_{\partial\Omega_{in}}\langle (\vec{q}\cdot\vec{n})v_\varepsilon(t,x), \xi(x)\rangle_{I_2}\, ds + \int_{\Omega^p_\varepsilon}\langle \vec{q}\cdot\nabla v_\varepsilon(t,x), \xi(x)\rangle_{I_2}\, dx$$
$$+\varepsilon\int_{\Gamma_\varepsilon}\left\langle \frac{\partial w_\varepsilon(t,x)}{\partial t}, \xi(x)\right\rangle_{I_2} d\sigma_x = \langle S_2R(u_\varepsilon(t), v_\varepsilon(t)), \xi\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^{I_2}\times[H^{1,q}(\Omega^p_\varepsilon)]^{I_2}}, \tag{3.2.4}$$

$$
\begin{aligned}
(iii) \quad & \int_{\Gamma_\varepsilon} \left\langle \frac{\partial w_\varepsilon(t)}{\partial t}, \varsigma \right\rangle_{I_2} d\sigma_x \;=\; -k_d \int_{\Gamma_\varepsilon} \langle z, \varsigma \rangle_{I_2} \, d\sigma_x, && (3.2.5)\\
(iv) \quad & z \;\in\; \psi(w_\varepsilon) \text{ on } (0,T) \times \Gamma_\varepsilon && (3.2.6)\\
& \text{for every } (\phi, \xi, \varsigma) \;\in\; [H^{1,q}(\Omega_\varepsilon^p)]^{I_1} \times [H^{1,q}(\Omega_\varepsilon^p)]^{I_2} \times [L^q(\Gamma_\varepsilon)]^{I_2} \text{ and for a.e. } t.\\
(v) \quad & (u_\varepsilon(0,x), v_\varepsilon(0,x), w_\varepsilon(0,x)) \;=\; (u_0(x), v_0(x), w_0(x)). && (3.2.7)
\end{aligned}
$$

3.3 Maximal Parabolic Regularity

Let $1 < p < \infty$, X be a Banach space and $A : D(A) \overset{d}{\subseteq} X \to X$ be a closed, not necessarily bounded, operator. Also assume that $f : (0,T) \to X$ is measurable. Consider the following problem

$$
\begin{aligned}
\frac{\partial u(t)}{\partial t} + Au(t) &= f(t) \quad \text{for } t > 0, && (3.3.1)\\
u(0) &= 0. && (3.3.2)
\end{aligned}
$$

In the theory of parabolic equations, it is well known that in general the time derivative, $\frac{\partial u}{\partial t}$, of the solution of (3.3.1)-(3.3.2) is less regular than f. One can look for a method so that this loss of regularity no longer occurs, i.e., for every $f \in L^p((0,T);X)$, does there exist a unique solution $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ of (3.3.1)-(3.3.2) which satisfies

$$
||u||_{L^p((0,T);X)} + ||u_t||_{L^p((0,T);X)} + ||u||_{L^p((0,T);D(A))} \;\le\; C\,||f||_{L^p((0,T);X)}, \tag{3.3.3}
$$

where the constant C is independent of u. The *maximal regularity property* of A resolves this issue. We start with the following definition.

Definition 3.3.1. *Let $1 < p < \infty$. The operator A is said to have the maximal (parabolic) L^p-regularity property if for every $f \in L^p((0,T);X)$, there exists a unique solution $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ of (3.3.1)-(3.3.2) which satisfies*

$$
||u||_{L^p((0,T);X)} + ||u_t||_{L^p((0,T);X)} + ||u||_{L^p((0,T);D(A))} \;\le\; C\,||f||_{L^p((0,T);X)}, \tag{3.3.4}
$$

where $C > 0$ is a constant.

For a detailed overview on maximal regularity, we refer the interested readers to [ACFP07], [Mon09], [Prü02], [RDR09], [KW04] and references therein.

3.3.1 Maximal Regularity of Differential Operators

Let $1 < p < \infty$. Set $D(A) := H^{1,p}(\Omega)$ and $X := H^{1,q}(\Omega)^*$. Clearly, $D(A) \overset{d}{\subseteq} X$. Let $\mu = (\mu_{ij})_{\substack{1 \le i \le n\\ 1 \le j \le n}}$ be a positive definite symmetric matrix-field, where $\mu_{ij} \in C(\bar{\Omega})$ and there is a constant $C > 0$

$$
\sum_{i,j=1}^{n} \mu_{ij}(x)\zeta_i\zeta_j \;\ge\; C|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^n \text{ and } x \in \Omega. \tag{3.3.5}
$$

We define a sesquilinear form $a(u,v) : H^{1,p}(\Omega) \times H^{1,q}(\Omega) \to \mathbb{R}$ by

$$
a(u,v) \;:=\; \int_\Omega \mu \nabla u \cdot \nabla v \, dx + \kappa \int_\Omega uv \, dx \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega), \tag{3.3.6}
$$

where $\kappa > 0$. We further define an operator $A : H^{1,p}(\Omega) \to H^{1,q}(\Omega)^*$ associated with the form $a(u,v)$ by

$$
\langle Au, v \rangle \;:=\; a(u,v) \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega). \tag{3.3.7}
$$

In [CL94] or [RDR09], it is shown that: (i) $||A^{is}||_{L(X)} \le Ke^{\theta|s|}$ for some $0 < \theta < \frac{\pi}{2}$, $s \in \mathbb{R}$, where $K > 0$ and (ii) $(-\infty, 0] \subset \rho(A)$ (resolvent of A) and $||(\lambda + A)^{-1}||_{L(X)} \le \frac{C}{1+|\lambda|}$ for every $\lambda \in [0,\infty)$, where $C > 0$. By theorem of Dore and Venni (cf. [DV87]),

$$\boxed{\text{the operator } A \text{ has maximal } L^p\text{-regularity on } H^{1,q}(\Omega)^*.} \tag{3.3.8}$$

Theorem 3.3.1 (Prüss and Schnaubelt). *Let $1 < p < \infty$ and $A : D(A) \overset{d}{\subseteq} X \to X$ be a closed linear operator with maximal L^p-regularity on X. Then for $u_0 \in (X, D(A))_{1-\frac{1}{p},p}$ and $f \in L^p((0,T);X)$, there exists a unique solution $u \in H^{1,p}((0,T);X) \cap L^p((0,T);D(A))$ of the problem*

$$\frac{\partial u(t)}{\partial t} + Au(t) = f(t) \quad \text{for } t > 0, \tag{3.3.9}$$

$$u(0) = u_0 \tag{3.3.10}$$

and we have the estimate

$$||u||_{L^p((0,T);D(A))} + ||u||_{L^p((0,T);X)} + \left|\left|\frac{\partial u}{\partial t}\right|\right|_{L^p((0,T);X)} \le C_p \left(||u_0||_{(X,D(A))_{1-\frac{1}{p},p}} + ||f||_{L^p((0,T);X)} \right),$$

where the constant C_p is independent of u_0, f and u.

Proof. See theorem 2.5 in [PS01]. ♦

3.4 Some Theorems and Lemmas

3.4.1 Trace Theorems

Lemma 3.4.1.1. *Let Γ_ε be as in (2.5.11). Then*

$$\varepsilon |\Gamma_\varepsilon| = |\Gamma| \frac{|\Omega|}{|Y|}. \tag{3.4.1}$$

Proof. [6]

$$\begin{aligned}
\varepsilon |\Gamma_\varepsilon| = \varepsilon \int_{\Gamma_\varepsilon} d\sigma_x = \varepsilon^n \sum_{k \in \mathbb{Z}^n} \int_{\Gamma_k} d\sigma_y = \varepsilon^n \sum_{k \in \mathbb{Z}^n} \frac{|\Gamma_k|}{|Y_k|} \int_{Y_k} dy &= \sum_{k \in \mathbb{Z}^n} \frac{|\Gamma_k|}{|Y_k|} \int_{\varepsilon Y_k} dx \\
&= \frac{|\Gamma|}{|Y|} \int_{\cup_{k \in \mathbb{Z}^n} \varepsilon Y^k} dx \\
&= \frac{|\Gamma|}{|Y|} \int_{\Omega} dx = \frac{|\Gamma|}{|Y|} |\Omega|.
\end{aligned}$$

♦

Theorem 3.4.1.2. *Let $1 \le p < \infty$. Suppose Y^p, Y^s and Γ are defined as in section 2.5.1. Then there exists a bounded linear operator $T : H^{1,p}(Y^p) \to L^p(\Gamma)$ such that*

$$(a)\ Tu := u|_\Gamma \text{ for } u \in H^{1,p}(Y^p) \cap C(\bar{Y}^p) \tag{3.4.2}$$

and

$$(b)\ ||u||^p_{L^p(\Gamma)} \le C_1 \left[||u||^p_{L^p(Y^p)} + ||\nabla u||^p_{L^p(Y^p)} \right], \tag{3.4.3}$$

where the constant C_1 depends on Y^p and p only.

[6] For $k \in \mathbb{Z}^n$, Γ_k and Y_k are the translated image of Γ and Y, $|\Gamma_k| = |\Gamma|$ and $|Y_k| = |Y|$. Also $x = \varepsilon y \Longrightarrow d\sigma_x = \varepsilon^{n-1} d\sigma_y$ and $dx = \varepsilon^n dy$.

Proof. See lemma 5.3 (a) in [HJ91]. See also lemma 2.7.2 in [NR92]. ♦

Theorem 3.4.1.3. *Let $1 \le p < \infty$. Let Ω_ε^p and Γ_ε be defined as in section 2.5.1. Then there exists a bounded linear operator $T^\varepsilon : H^{1,p}(\Omega_\varepsilon^p) \to L^p(\Gamma_\varepsilon)$ such that*

$$(a)\, T^\varepsilon u \;=\; u|_{\Gamma_\varepsilon} \text{ for } u \in H^{1,p}(\Omega_\varepsilon^p) \cap C(\bar{\Omega}_\varepsilon^p) \tag{3.4.4}$$

and

$$(b)\, \varepsilon \int_{\Gamma_\varepsilon} |T^\varepsilon u(x)|^p \, d\sigma_x \;\le C_2 \left(\int_{\Omega_\varepsilon^p} |u(x)|^p \, dx + \varepsilon^p \int_{\Omega_\varepsilon^p} |\nabla_x u(x)|^p \, dx \right), \tag{3.4.5}$$

where the constant C_2 is independent of ε and u.

Proof. The proof follows by a scaling argument. For details confer lemma 5.3 (b) in [HJ91]. See also lemma 2.7.2 in [NR92]. ♦

3.4.2 Extension Theorems

Lemma 3.4.2.1. *Let $1 \le p \le \infty$. For $u \in H^{1,p}(Y^p)$, there exists an extension $\tilde{u}$ of u into all of Y such that*

$$(a)\, \tilde{u} \;:=\; u \text{ in } Y^p \tag{3.4.6}$$

and

$$(b)\, ||\tilde{u}||^p_{H^{1,p}(Y)} \;\le\; C_3 ||u||^p_{H^{1,p}(Y^p)}, \tag{3.4.7}$$

where the constant C_3 depends on p and Y^p only but is independent of u and $\tilde{u}$.

Proof. Confer lemma 5 (a) in [HJ91]. ♦

Theorem 3.4.2.2. *Let $1 \le p \le \infty$. Suppose that Ω_ε^p and Ω are defined as in section 2.5.1. For $u \in H^{1,p}(\Omega_\varepsilon^p)$, there exists a bounded linear operator $P^\varepsilon : H^{1,p}(\Omega_\varepsilon^p) \to H^{1,p}(\Omega)$ such that*

$$(a)\, P^\varepsilon u \;:= u \quad \text{in } \Omega_\varepsilon^p \tag{3.4.8}$$

and

$$(b)\, ||P^\varepsilon u||^p_{H^{1,p}(\Omega)} \;\le\; C_4 ||u||^p_{H^{1,p}(\Omega_\varepsilon^p)}, \tag{3.4.9}$$

where the constant C_4 is independent of ε and u but depends on p.

Proof. The proof follows by a scaling argument[7]. For details confer theorem 5.2 in [HJ91]. See also [Tar80]. ♦

Now we prove a theorem similar to theorem 3.4.2.2 for the functions depending on both t and x. Let $1 \le p \le \infty$. For $u \in L^p((0,T); H^{1,p}(\Omega_\varepsilon^p))$, we define an operator $Q^\varepsilon : L^p((0,T); H^{1,p}(\Omega_\varepsilon^p)) \to L^p((0,T); H^{1,p}(\Omega))$ such that

$$Q^\varepsilon u(t,x) \;\;:=\;\; [P^\varepsilon u(t,.)]\,(x) \quad \text{for } u \in L^p((0,T); H^{1,p}(\Omega_\varepsilon^p)), \tag{3.4.10}$$

where P^ε is the extension operator from theorem 3.4.2.2. Then

$$\frac{\partial}{\partial t}[Q^\varepsilon u(t,x)] \;=\; \frac{\partial}{\partial t}[P^\varepsilon u(t,.)]\,(x) \;=\; \left[P^\varepsilon(\frac{\partial u}{\partial t}(t,.)) \right](x) \;=\; Q^\varepsilon(\frac{\partial u}{\partial t})(t,x).$$

Based on the above definition we have the following extension theorem for the functions depending on t and x.

[7] A more general form of this theorem is given in Miller [Mil92].

Theorem 3.4.2.3. *Let Ω and Ω_ε^p be defined as in section 2.5.1 and $1 \leq p \leq \infty$. Then there exists a bounded linear operator $Q^\varepsilon : L^p((0,T);H^{1,p}(\Omega_\varepsilon^p)) \cap H^{1,p}((0,T);L^p(\Omega_\varepsilon^p)) \to L^p((0,T);H^{1,p}(\Omega)) \cap H^{1,p}((0,T);L^p(\Omega))$ such that for all $u \in L^p((0,T);H^{1,p}(\Omega_\varepsilon^p)) \cap H^{1,p}((0,T);L^p(\Omega_\varepsilon^p))$*

$$(a)\ Q^\varepsilon(\frac{\partial u}{\partial t}) \;=\; \frac{\partial}{\partial t}(Q^\varepsilon u(t)) \tag{3.4.11}$$

and

$$(b)\ ||Q^\varepsilon u||_{L^p((0,T);H^{1,p}(\Omega))} \;\leq\; C_5\, ||u||_{L^p((0,T);H^{1,p}(\Omega_\varepsilon^p))}\,, \tag{3.4.12}$$

where the constant C_5 is independent of ε and u.

Proof. Here we only show the measurability of $Q^\varepsilon u$. The inequality (3.4.12) follows by scaling. Since we know that every continuous function is measurable, we show $Q^\varepsilon u$ is continuous. But by theorem 3.4.2.2 it can be shown $P^\varepsilon u(t)$ is continuous on $\bar{\Omega}$. The continuity of $Q^\varepsilon u$ on $[0,T] \times \bar{\Omega}$ follows from the definition (3.4.10). ♦

Theorem 3.4.2.4. *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in (H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}$. Then there exists an extension $\bar{u}$ of u such that $\bar{u} \in (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$.*

Proof. Let $\theta = 1 - \frac{1}{p}$. We use the K-functional definition for real interpolation space $(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{\theta,p}$.[8] To begin with, let $v \in H^{1,q}(\Omega_\varepsilon^p)$, then by theorem 3.4.2.2 there exists an extension $P^\varepsilon v$ of v such that

$$(a)\ P^\varepsilon v \;:=\; v \text{ in } \Omega_\varepsilon^p \tag{3.4.13}$$
$$\text{and} \quad (b)\ ||P^\varepsilon v||_{H^{1,q}(\Omega)} \;\leq\; C_4 ||v||_{H^{1,q}(\Omega_\varepsilon^p)}\,, \tag{3.4.14}$$

where C_4 is independent of ε and v. Let $a_0 \in H^{1,q}(\Omega_\varepsilon^p)^*$. Since P^ε is a linear operator from $H^{1,q}(\Omega_\varepsilon^p)$ into $H^{1,q}(\Omega)$, we can define a function $\bar{a}_0$ (an extension of a_0) by the following formula

$$\langle \bar{a}_0, P^\varepsilon v\rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} := \langle a_0, v\rangle_{H^{1,q}(\Omega_\varepsilon^p)^* \times H^{1,q}(\Omega_\varepsilon^p)} \text{ for any } v \in H^{1,q}(\Omega_\varepsilon^p). \tag{3.4.15}$$

Therefore

$$\begin{aligned}
||\bar{a}_0||_{H^{1,q}(\Omega)^*} &= \sup_{||P^\varepsilon v||_{H^{1,q}(\Omega)} \leq 1} \left|\langle \bar{a}_0, P^\varepsilon v\rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)}\right| \\
&= \sup_{||v||_{H^{1,q}(\Omega_\varepsilon^p)} \leq 1} \left|\langle a_0, v\rangle_{H^{1,q}(\Omega_\varepsilon^p)^* \times H^{1,q}(\Omega_\varepsilon^p)}\right| \text{ by (3.4.14) and (3.4.15)} \\
&\leq ||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*} \\
\Longrightarrow ||\bar{a}_0||_{H^{1,q}(\Omega)^*} &\leq ||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*}\,. \qquad (3.4.16)
\end{aligned}$$

Again assume that $b_0 \in H^{1,p}(\Omega_\varepsilon^p)$. Let $\bar{b}_0 \in H^{1,p}(\Omega)$ denote the extension of b_0 such that

$$\left|\left|\bar{b}_0\right|\right|_{H^{1,p}(\Omega)} \;\leq\; C_4 ||b_0||_{H^{1,p}(\Omega_\varepsilon^p)} \text{ for } b_0 \in H^{1,p}(\Omega_\varepsilon^p), \tag{3.4.17}$$

where C_4 is independent of ε and b_0. Let $t > 0$. Then

$$\begin{aligned}
||\bar{a}_0||_{H^{1,q}(\Omega)^*} + t\left|\left|\bar{b}_0\right|\right|_{H^{1,p}(\Omega)} &\leq ||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*} + C_4 t ||b_0||_{H^{1,p}(\Omega_\varepsilon^p)} \\
&\leq \max(1, C_4)\left(||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*} + t||b_0||_{H^{1,p}(\Omega_\varepsilon^p)}\right).
\end{aligned}$$

[8] For the definition of real-interpolation space see [Lun95], [Tri95], [BL76].

Taking the infimum on both sides, we get

$$\begin{aligned}
&\inf_{\substack{\bar u=\bar a_0+\bar b_0\\ \bar a_0\in H^{1,q}(\Omega)^*\\ \bar b_0\in H^{1,p}(\Omega)}}\left(||\bar a_0||_{H^{1,q}(\Omega)^*}+t\left|\left|\bar b_0\right|\right|_{H^{1,p}(\Omega)}\right)\\
&\qquad\le\ \max(1,C_4)\inf_{\substack{u=a_0+b_0\\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}}\left(||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*}+t\,||b_0||_{H^{1,p}(\Omega_\varepsilon^p)}\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\underbrace{t^{-\theta}\inf_{\substack{\bar u=\bar a_0+\bar b_0\\ \bar a_0\in H^{1,q}(\Omega)^*\\ \bar b_0\in H^{1,p}(\Omega)}}\left(||\bar a_0||_{H^{1,q}(\Omega)^*}+t\left|\left|\bar b_0\right|\right|_{H^{1,p}(\Omega)}\right)}_{positive}\\
&\qquad\le\ \max(1,C_4)\,\underbrace{t^{-\theta}\inf_{\substack{u=a_0+b_0\\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}}\left(||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*}+t\,||b_0||_{H^{1,p}(\Omega_\varepsilon^p)}\right)}_{positive},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left|t^{-\theta}\inf_{\substack{\bar u=\bar a_0+\bar b_0\\ \bar a_0\in H^{1,q}(\Omega)^*\\ \bar b_0\in H^{1,p}(\Omega)}}\left(||\bar a_0||_{H^{1,q}(\Omega)^*}+t\left|\left|\bar b_0\right|\right|_{H^{1,p}(\Omega)}\right)\right|^p\\
&\le\ [\max(1,C_4)]^p\left|t^{-\theta}\inf_{\substack{u=a_0+b_0\\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}}\left(||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*}+t\,||b_0||_{H^{1,p}(\Omega_\varepsilon^p)}\right)\right|^p.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^\infty\left|t^{-\theta}\inf_{\substack{\bar u=\bar a_0+\bar b_0\\ \bar a_0\in H^{1,q}(\Omega)^*\\ \bar b_0\in H^{1,p}(\Omega)}}\left(||\bar a_0||_{H^{1,q}(\Omega)^*}+t\left|\left|\bar b_0\right|\right|_{H^{1,p}(\Omega)}\right)\right|^p\frac{dt}{t}\\
&\quad\le\ [\max(1,C_4)]^p\int_0^\infty\left|t^{-\theta}\inf_{\substack{u=a_0+b_0\\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}}\left(||a_0||_{H^{1,q}(\Omega_\varepsilon^p)^*}+t\,||b_0||_{H^{1,p}(\Omega_\varepsilon^p)}\right)\right|^p\frac{dt}{t},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_0^\infty\left|t^{-\theta}K(t,\bar u,H^{1,q}(\Omega)^*,H^{1,p}(\Omega))\right|^p\frac{dt}{t}\\
&\quad\le\ [\max(1,C_4)]^p\int_0^\infty\left|t^{-\theta}K(t,u,H^{1,q}(\Omega_\varepsilon^p)^*,H^{1,p}(\Omega_\varepsilon^p))\right|^p\frac{dt}{t},
\end{aligned}$$

i.e.,

$$\begin{aligned} &||\bar{u}||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \\ &\quad \leq \max(1, C_4)\, ||u||_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}} = C_6\, ||u||_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}}, \end{aligned} \tag{3.4.18}$$

where the constant C_6 $(:= \max(1, C_4))$ is independent of ε and u. ♦

3.4.3 Embedding Theorems

Theorem 3.4.3.1. *Let Ω and Ω_ε^p be as in section 2.5.1. Assume that $1 \leq p < n$ and $u \in H^{1,p}(\Omega_\varepsilon^p)$. Then $u \in L^q(\Omega_\varepsilon^p)$ and there is a constant C_7*

$$||u||_{L^q(\Omega_\varepsilon^p)} \leq C_7\, ||u||_{H^{1,p}(\Omega_\varepsilon^p)}, \tag{3.4.19}$$

where $q = \frac{np}{n-p}$ and C_7 is independent of ε and u.
In other words, $H^{1,p}(\Omega_\varepsilon^p) \hookrightarrow L^q(\Omega_\varepsilon^p)$ with embedding constant independent of ε.

Proof. Let $u \in H^{1,p}(\Omega_\varepsilon^p)$. Then from theorem 3.4.2.2, there exists an extension $P^\varepsilon u$ of u from $H^{1,p}(\Omega_\varepsilon^p)$ to $H^{1,p}(\Omega)$ such that

$$||P^\varepsilon u||_{H^{1,p}(\Omega)} \leq C_4\, ||u||_{H^{1,p}(\Omega_\varepsilon^p)}. \tag{3.4.20}$$

Let $v := P^\varepsilon u$. By assumption Ω is a bounded domain with sufficiently smooth boundary, then from theorem 2 of section 5.6.1 in [Eva98] we get

$$||v||_{L^q(\Omega)} \leq C\, ||v||_{H^{1,p}(\Omega)} \text{ for } v \in H^{1,p}(\Omega), \tag{3.4.21}$$

where $q = \frac{np}{n-p}$ and C depends only on p, n and Ω but is independent of v.
Therefore

$$\begin{aligned} ||u||^q_{L^q(\Omega_\varepsilon^p)} &= \int_{\Omega_\varepsilon^p} |u(x)|^q\, dx = \int_{\Omega_\varepsilon^p} |v(x)|^q\, dx \\ &\leq \int_\Omega |v(x)|^q\, dx \\ &\leq C^q\, ||v||^q_{H^{1,p}(\Omega)} \text{ from (3.4.21)} \\ &= C^q\, ||P^\varepsilon u||^q_{H^{1,p}(\Omega)} \\ &\leq C^q\, C_4^q\, ||u||^q_{H^{1,p}(\Omega_\varepsilon^p)} \text{ from (3.4.20)} \\ \Longrightarrow ||u||_{L^q(\Omega_\varepsilon^p)} &\leq C_7\, ||u||_{H^{1,p}(\Omega_\varepsilon^p)}, \end{aligned}$$

where C_7 $(:= C\, C_4)$ is independent of ε and u. ♦

Theorem 3.4.3.2. *Suppose that Ω and Ω_ε^p are as in section 2.5.1. Then for $u \in H^{1,2}(\Omega_\varepsilon^p)$ the following inequality holds*

$$||u||^2_{L^2(\partial\Omega)} \leq C_8\, ||u||_{H^{1,2}(\Omega_\varepsilon^p)}\, ||u||_{L^2(\Omega_\varepsilon^p)}, \tag{3.4.22}$$

where the constant C_8 is independent of ε and u.

Proof. The proof follows by combining the theorems 3.4.2.2 and B.7. ♦

Theorem 3.4.3.3. *Let $1 < p,q < \infty$ be such that $p > n+2$ and $\frac{1}{p}+\frac{1}{q} = 1$. Assume that $u \in (H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}$ such that $\sup_{\varepsilon>0} ||u||_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}} < \infty$. Then $u \in L^\infty(\Omega_\varepsilon^p)$ and*

$$\sup_{\varepsilon>0} ||u||_{L^\infty(\Omega_\varepsilon^p)} < \infty. \tag{3.4.23}$$

Proof. From theorem 3.1.3, we know that for $u \in (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$, $u \in L^\infty(\Omega)$ and

$$||u||_{L^\infty(\Omega)} \leq C_9 ||u||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}, \tag{3.4.24}$$

where the constant C_9 is independent of u. Let $u \in (H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}$, then

$$\begin{aligned}
||u||_{L^\infty(\Omega_\varepsilon^p)} &= \operatorname*{ess\,sup}_{x\in\Omega_\varepsilon^p} |u(x)| \\
&\leq \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \\
&= ||u||_{L^\infty(\Omega)} \\
&\leq C_9 ||u||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \qquad \text{by (3.4.24)} \\
&\leq C_6 C_9 ||u||_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}} \qquad \text{by theorem 3.4.2.4} \\
&\leq C_6 C_9 \sup_{\varepsilon>0} ||u||_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}} \\
&< \infty \quad \forall \varepsilon > 0,
\end{aligned}$$

where the constants C_6 and C_9 are independent of ε and u. Therefore $\sup_{\varepsilon>0} ||u||_{L^\infty(\Omega_\varepsilon^p)} < \infty$. ♦

Theorem 3.4.3.4. *Let $p > n+2$, then $\mathcal{F}_\varepsilon^u \hookrightarrow\hookrightarrow [L^\infty((0,T)\times\Omega_\varepsilon^p)]^I$.*

Proof. Since $F_\varepsilon := H^{1,p}((0,T); H^{1,q}(\Omega_\varepsilon^p)^*) \cap L^p((0,T); H^{1,p}(\Omega_\varepsilon^p)) \hookrightarrow\hookrightarrow L^\infty((0,T)\times\Omega_\varepsilon^p)$ by theorem 3.1.2. Therefore the theorem follows. ♦

3.5 Two-scale Convergence

Definition 3.5.1. *Let ε be a sequence of positive real numbers converging to 0. A sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ in $L^p(\Omega)$ is said to two-scale convergent to a limit $u \in L^p(\Omega\times Y)$ if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\phi(x,\frac{x}{\varepsilon})\,dx = \int_\Omega\int_Y u(x,y)\phi(x,y)\,dx\,dy, \tag{3.5.1}$$

for all $\phi \in L^q(\Omega; C_{per}(Y))$.[9]

The above definition is followed from the following theorem which is proved by Nguetseng in [Ngu89].

Theorem 3.5.3. *For every bounded sequence, $(u_\varepsilon)_{\varepsilon>0}$, in $L^p(\Omega)$ there exist a subsequence and a $u \in L^p(\Omega\times Y)$ such that the subsequence two-scale converges to u.*

Proof. See theorem 1 in [Ngu89]. Confer also theorem 14 in [LNW02] or theorem 0.1 in [All92]. ♦

Remark 3.5.4. If $(u_\varepsilon)_{\varepsilon>0}$ is two-scale convergent to u then we write $u_\varepsilon \xrightarrow{2} u$.

[9] $C_{per}(Y)$ denotes the space of Y-periodic continuous functions in y.

We state some theorems on two-scale convergence. The proofs of all these theorems can be found in [Ngu89], [LNW02], [All92].

Theorem 3.5.5. *Let $(u_\varepsilon)_{\varepsilon>0}$ be strongly convergent to $u \in L^p(\Omega)$, then $(u_\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $u_1(x,y) = u(x)$.*

Proof. Cf. theorem 9 in [LNW02]. ♦

Theorem 3.5.6. *Let $(u_\varepsilon)_{\varepsilon>0}$ be two-scale convergent to u in $L^p(\Omega\times Y)$, then $(u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to $\int_Y u(x,y)\,dy$ in $L^p(\Omega)$ and $(u_\varepsilon)_{\varepsilon>0}$ is bounded.*

Proof. Cf. theorem 19 in [LNW02]. ♦

In the defintion 3.5.1, one can notice that the space of test functions is chosen as $L^q(\Omega; C_{per}(Y))$, but we can replace the space of test functions by $C_0^\infty(\Omega; C_{per}^\infty(Y))$, if $(u_\varepsilon)_{\varepsilon>0}$ satisfies certain condition which is given in the following theorem:

Theorem 3.5.7. *Let $(u_\varepsilon)_{\varepsilon>0}$ be bounded in $L^p(\Omega)$ such that*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\phi(x,\frac{x}{\varepsilon})\,dx = \int_\Omega\int_Y u(x,y)\phi(x,y)\,dx\,dy \text{ for all } \phi \in C_0^\infty(\Omega; C_{per}^\infty(Y)). \tag{3.5.2}$$

Then $(u_\varepsilon)_{\varepsilon>0}$ is two-scale convergent to u.

Proof. See proposition 13 in [LNW02]. ♦

Theorem 3.5.8. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $H^{1,p}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ in $H^{1,p}(\Omega)$. Then $(u_\varepsilon)_{\varepsilon>0}$ two-scale converges to u and there exist a subsequence ε, still denoted by same symbol, and a $u_1 \in L^p(\Omega; H_{per}^{1,p}(Y))$ such that $\nabla_x u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \nabla_y u_1$.*

Proof. Cf. theorem 20 in [LNW02]. ♦

Theorem 3.5.9. *Let $(u_\varepsilon)_{\varepsilon>0}$ and $(\varepsilon\nabla_x u_\varepsilon)_{\varepsilon>0}$ be bounded in $L^p(\Omega)$ and $[L^p(\Omega)]^n$ respectively. Then there exists $u \in L^p(\Omega; H_{per}^{1,p}(Y))$ such that up to a subsequence, still denoted by ε, we have*

$$u_\varepsilon \overset{2}{\rightharpoonup} u$$

$$\textit{and} \quad \varepsilon\nabla_x u_\varepsilon \overset{2}{\rightharpoonup} \nabla_y u$$

as $\varepsilon \to 0$.

Proof. Cf. theorem 3.16 in [Zie09]. ♦

Since in this work we will only consider evolution equations which introduces time as an additional parameter, we transfer the definition 3.5.1 to the functions depending on t and x.

Definition 3.5.10. *A sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ in $L^p((0,T)\times\Omega)$ is said to two-scale convergent to a limit $u \in L^p((0,T)\times\Omega\times Y)$ if*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon})\,dx\,dt = \int_0^T\int_\Omega\int_Y u(t,x,y)\phi(t,x,y)\,dx\,dy\,dt \tag{3.5.3}$$

for all $\phi \in L^q((0,T)\times\Omega; C_{per}(Y))$.

All the above theorems on two-scale convergence can be generalized for the functions depending on t and x. Here we only give the statements of such theorems. For the proof of these theorems, see in [Cla98], [Pet03], [Zie09], [NR92].

Theorem 3.5.11. *For every bounded sequence, $(u_\varepsilon)_{\varepsilon>0}$, in $L^p((0,T)\times\Omega)$ there exist a subsequence and a $u \in L^p((0,T)\times\Omega\times Y)$ such that the subsequence two-scale converges to u.*

Theorem 3.5.12. *Let $(u_\varepsilon)_{\varepsilon>0}$ be strongly convergent to $u \in L^p((0,T)\times\Omega)$, then $(u_\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $u_1(t,x,y)=u(t,x)$.*

Theorem 3.5.13. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p((0,T);H^{1,p}(\Omega))$ such that $u_\varepsilon \to u$ weakly in $L^p((0,T);H^{1,p}(\Omega))$. Then $(u_\varepsilon)_{\varepsilon>0}$ two-scale converges to u and there exist a subsequence ε, still denoted by same symbol, and a $u_1 \in L^p((0,T)\times\Omega;H^{1,p}_{per}(Y))$ such that $\nabla_x u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \nabla_y u_1$.*

Theorem 3.5.14. *Let $(u_\varepsilon)_{\varepsilon>0}$ and $(\varepsilon\nabla_x u_\varepsilon)_{\varepsilon>0}$ be bounded in $L^p((0,T)\times\Omega)$ and $[L^p((0,T)\times\Omega)]^n$ respectively. Then there exists $u \in L^p((0,T)\times\Omega;H^{1,p}_{per}(Y))$ such that up to a subsequence, still denoted by ε, we have*

$$u_\varepsilon \overset{2}{\rightharpoonup} u$$

$$\text{and} \quad \varepsilon\nabla_x u_\varepsilon \overset{2}{\rightharpoonup} \nabla_y u$$

as $\varepsilon \to 0$.

Next we define the notion of two-scale convergence on the $(n-1)$ dimensional surface Γ_ε. We follow the notations of section 2.5.1.

Definition 3.5.15 (cf. [ADH96], [NR96]). *Let $1 \le p < \infty$. A sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^p((0,T)\times\Gamma_\varepsilon)$ is said to two-scale convergent to a limit $u \in L^p((0,T)\times\Omega\times\Gamma)$ if*

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} u_\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt = \int_0^T \int_\Omega \int_\Gamma u(t,x,y)\phi(t,x,y)\,dx\,dy\,dt \tag{3.5.4}$$

for all $\phi \in C([0,T]\times\bar{\Omega};C_{per}(Y))$.

Theorem 3.5.16. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p((0,T)\times\Gamma_\varepsilon)$ such that*

$$\varepsilon \int_0^T \int_{\Gamma_\varepsilon} |u_\varepsilon(t,x)|^p\,d\sigma_x\,dt \le C, \tag{3.5.5}$$

where C is independent of ε. Then there exists a subsequence (still denoted by ε) and a two-scale limit $u \in L^p((0,T)\times\Omega\times\Gamma)$ such that u_ε is two-scale convergent to u in the sense of (3.5.4).

Proof. Confer theorem 2.1 in [ADH96]. ♦

3.6 Periodic Unfolding

Arbogast, Douglas, and Hornung in [ADH90] introduced the concept of *dilation operator* to study the homogenization on periodic domains with double porosity. This method is further used in [BLM96], [NRJ07], [ACP08] etc. Later on the idea of dilation operator is extended by Cioranescu, Damlamian and Griso (cf. [CDG02], [CDG08]) to examine the homogenization problems on periodic domains under the name of *periodic unfolding*. We continue our discussion with the definition of periodic unfolding on fixed domains.

Let Ω, Y, m, k and Γ_ε be defined as in section 2.5.1. For any $z \in \mathbb{R}^n$, suppose $[z]$ denotes the unique integer combination $\sum_{j=i}^n k_j e_j$ of e_j such that $z-[z]$ lies in Y (see figure 3.6.1) and we set

$$\{z\} = z - [z] \quad \text{for a.e. } z \in \mathbb{R}^n.$$

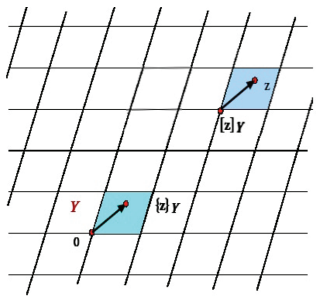


Figure 3.6.1: Definition of $[z]$ and $\{z\}$.[10]

Thus for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we have

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right] + \left\{\frac{x}{\varepsilon}\right\}\right) \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{3.6.1}$$

Setting

$$\begin{aligned} \Xi_\varepsilon &= \{\xi \in \mathbb{Z}^n : \varepsilon(\xi + Y) \subset \Omega\}, \\ \hat{\Omega}_\varepsilon &= \ \mathrm{int}\left\{\cup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y})\right\} \end{aligned}$$

and

$$\Lambda_\varepsilon = \Omega - \hat{\Omega}_\varepsilon.$$

Definition 3.6.1. *Assume that $1 \le p \le \infty$. Let $u \in L^p((0,T)\times\Omega)$ such that for every t, $u(t)$ is extended by zero outside of Ω. We define the unfolding operator $T^\varepsilon : L^p((0,T)\times\Omega) \to L^p((0,T)\times\Omega\times Y)$ as*

$$\begin{aligned} T^\varepsilon u(t,x,y) &= u\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) && \text{for a.e. } (t,x,y) \in (0,T)\times\hat{\Omega}_\varepsilon\times Y, \\ &= 0 && \text{for a.e. } (t,x,y) \in (0,T)\times\Lambda_\varepsilon\times Y. \end{aligned} \tag{3.6.2}$$

We collect some properties of T^ε.

Theorem 3.6.2. *Let $1 < p < \infty$. Then the unfolding operator T^ε has the following properties:*

(i) T^ε *is linear.*

(ii) *If $u \in L^p((0,T)\times\Omega)$, then for a.e. t and x, $T^\varepsilon u(t,x,\{\frac{x}{\varepsilon}\}) = u(t,x)$.*

(iii) *Let $u, v \in L^p((0,T)\times\Omega)$, then $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.*

(iv) *Let $u \in L^1((0,T)\times\Omega)$, then $\int_0^T \int_{\hat{\Omega}_\varepsilon} u(t,x)\,dt\,dx = \frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} T^\varepsilon(u(t,x,y))\,dx\,dy\,dt$.*

(v) *Let $u \in L^p((0,T)\times\Omega)$, then $||T^\varepsilon u||_{L^p((0,T)\times\Omega\times Y)} \le |Y|^{\frac{1}{p}}\,||u||_{L^p((0,T)\times\Omega)}$.*

(vi) *Let $u \in L^p((0,T)\times\Omega)$, then $(T^\varepsilon u)_{\varepsilon>0}$ is strongly convergent to u in $L^p((0,T)\times\Omega\times Y)$.*

(vii) *Let $(u_\varepsilon)_{\varepsilon>0} \subset L^p((0,T)\times\Omega)$ be such that $(T^\varepsilon u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to $\tilde{u}$ in $L^p((0,T)\times\Omega\times Y)$, then $(u_\varepsilon)_{\varepsilon>0}$ weakly converges to u in $L^p((0,T)\times\Omega)$, where $u = \frac{1}{|Y|}\int_Y \tilde{u}\,dy$.*

[10]This figure is provided to the author by Prof. Alan Damlamian via personal communication.

(viii) Let $(u_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^p((0,T)\times\Omega)$. Then the following statements are equivalent:

(a) $(T^\varepsilon(u_\varepsilon))_{\varepsilon>0}$ weakly converges to u in $L^p((0,T)\times\Omega\times Y)$.

(b) $(u_\varepsilon)_{\varepsilon>0}$ two-scale converges to u.

Proof. For the proofs of (i)-(viii), confer [CDG02], [CDG08] and [CDZ06]. ♦

Next we define the concept of *boundary unfolding operator* on Γ_ε (cf. [CDZ06]).

Definition 3.6.3. *Let $1\le p\le\infty$. For any $u\in L^p((0,T)\times\Gamma_\varepsilon)$, the boundary unfolding operator $T_b^\varepsilon : L^p((0,T)\times\Gamma_\varepsilon)\to L^p((0,T)\times\Omega\times\Gamma)$ is defined as*

$$T_b^\varepsilon u(t,x,y) := u\left(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y\right) \quad \text{for every } (t,x,y)\in(0,T)\times\Omega\times\Gamma. \tag{3.6.3}$$

Theorem 3.6.4. *Let $1<p<\infty$. Then the operator T_b^ε has the following properties:*

(i) T_b^ε is linear.

(ii) If $u\in L^p((0,T)\times\Gamma_\varepsilon)$, then $T_b^\varepsilon u(t,x,\{\frac{x}{\varepsilon}\}) = u(t,x)$, for every *$t\in(0,T)$ and $x\in\Omega$.*

(iii) Let $u,v\in L^p((0,T)\times\Gamma_\varepsilon)$, then $T_b^\varepsilon(uv) = T_b^\varepsilon(u)T_b^\varepsilon(v)$.

(iv) Let $u\in L^1((0,T)\times\Gamma_\varepsilon)$, then $\int_0^T\int_{\Gamma_\varepsilon} u(t,x)\,dt\,d\sigma_x = \frac{1}{\varepsilon|Y|}\int_0^T\int_{\Omega\times Y} T_b^\varepsilon(u(t,x,y))\,dx\,d\sigma_y\,dt$.

(v) Let $u_\varepsilon\in L^p((0,T)\times\Gamma_\varepsilon)$, then $\int_0^T\int_\Omega\int_\Gamma |T_b^\varepsilon u(t,x,y)|^p\,dx\,dy\,dt = \varepsilon|Y|\int_0^T\int_{\Gamma_\varepsilon}|u(t,x)|^p\,d\sigma_x\,dt$.

(vi) Let $u\in L^p((0,T)\times\Omega)$, then $(T_b^\varepsilon u)_{\varepsilon>0}$ is strongly convergent to u in $L^p((0,T)\times\Omega\times\Gamma)$.

(vii) Let $(u_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^p((0,T)\times\Gamma_\varepsilon)$. Then the following statements are equivalent:

(a) $(T_b^\varepsilon(u_\varepsilon))_{\varepsilon>0}$ weakly converges to u in $L^p((0,T)\times\Omega\times\Gamma)$.

(b) $(u_\varepsilon)_{\varepsilon>0}$ two-scale converges to u in the sense of (3.5.4).

Proof. The proofs of (i)-(vii) can be found in [CDG02], [CDG08] and [CDZ06]. ♦

Existence of a Unique Positive Global Weak Solution of a System of Diffusion – Reaction Equations and Homogenization

This chapter is the main body of this work and investigates the models I and II, introduced in chapter 2. Section 4.1 deals with the model M1. In section 4.1.1 we prove the positivity, existence and uniqueness of the solution of the problem (P_ε^1) which is global in time. We obtain some ε-independent *a-priori* estimates of this solution in section 4.1.2.1 and we upscale the model from the micro scale to the macro scale in section 4.1.2.3. Next we treat the model M2 in section 4.2. We discuss the positivity, existence and uniqueness of the global solution of (P_ε^2) in section 4.2.1. In section 4.2.2.1 some ε-independent *a-priori* estimates are obtained. Finally we conclude this chapter with the homogenization of model M2.

4.1 Model M1

4.1.1 Existence and Uniqueness of the Global Solution of (P_ε^1)

Let the following assumptions be satisfied:[11]

$$(i)\ p > n+2. \tag{4.1.1}$$

$$(ii)\ u_0 \geq 0,\ \text{i.e., } u_{0_i} \geq 0 \text{ for all } i = 1,2,...,I. \tag{4.1.2}$$

$$(iii)\ u_{0_i} \in (H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p} \text{ for } i = 1,2,...,I. \tag{4.1.3}$$

(iv) All reactions are linearly independent such that the stoichiometric matrix

$$S = (s_{ij})_{\substack{1\leq i\leq I\\ 1\leq j\leq J}} \text{ has maximal column rank, i.e., } rank(S) = J. \tag{4.1.4}$$

$$(v)\ \sup_{\varepsilon>0} ||u_{0_i}||_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}} < \infty \text{ for all } i = 1,2,...,I. \tag{4.1.5}$$

Theorem 4.1.1.1 (Existence theorem). *Suppose that the assumptions (4.1.1)-(4.1.5) are satisfied, then there exists a unique positive global weak solution $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ of the problem (P_ε^1).*

Strategy of the proof: We adopt the methodology of Kräutle (cf. [Krä08], [Krä11]) in order to prove the positivity, existence and uniqueness of the global solution of the problem (P_ε^1). As already mentioned in chapter 1 that for $p > n+1$, on the macroscopic level, Kräutle has shown the existence of a unique positive global weak solution in $[H^{1,p}((0,T);L^p(\Omega)) \cap L^p(0,T);H^{2,p}(\Omega))]^I$ for the problem (2.5.16)-(2.5.19). Here we will show that with a little stronger condition on p, i.e., for $p > n+2$, there exists a unique

[11] Note that in case of model M1, we only have I number of mobile species. We choose $I_1 = I$ in the function spaces introduced in section 3.1.

positive global weak solution of the problem (P_ε^1) in $\mathcal{F}_\varepsilon^u$.

Before dealing with the problem (P_ε^1), we consider a slightly modified problem and introduce the rate function $\bar{R}:\mathbb{R}^I \to \mathbb{R}^J$ as

$$\bar{R}(u_\varepsilon) := R(u_\varepsilon^+), \tag{4.1.6}$$

where u_ε^+ is the positive part of u_ε defined componentwise as

$$\left.\begin{aligned} u_{\varepsilon_i}^+ &:= \max(u_{\varepsilon_i},0),\\ u_{\varepsilon_i}^- &:= \max(-u_{\varepsilon_i},0) = -\min(u_{\varepsilon_i},0)\\ \text{and}\qquad u_{\varepsilon_i} &= u_{\varepsilon_i}^+ - u_{\varepsilon_i}^-. \end{aligned}\right\} \tag{4.1.7}$$

This gives

$$\boxed{\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \nabla\cdot D\nabla u_\varepsilon = S\bar{R}(u_\varepsilon) \qquad &\text{in} \quad (0,T)\times\Omega_\varepsilon^p, &(4.1.8)\\ u_\varepsilon(0,x) = u_0(x) \qquad &\text{in} \quad \Omega_\varepsilon^p, &(4.1.9)\\ -D\nabla u_\varepsilon\cdot\vec{n} = 0 \qquad &\text{on} \quad (0,T)\times\partial\Omega, &(4.1.10)\\ -D\nabla u_\varepsilon\cdot\vec{n} = 0 \qquad &\text{on} \quad (0,T)\times\Gamma_\varepsilon. &(4.1.11) \end{aligned}}$$

Let us denote this problem by $(P_\varepsilon^{1^+})$. We will prove the existence of a global solution of $(P_\varepsilon^{1^+})$. Since we show that the solution of $(P_\varepsilon^{1^+})$ is non-negative, it solves (P_ε^1). We conclude this section by proving the uniqueness of the solution of (P_ε^1). We commence our investigation of the positivity of the solution of $(P_\varepsilon^{1^+})$.

Lemma 4.1.1.2. *Let (4.1.1)-(4.1.5) hold and a function $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ be the solution of $(P_\varepsilon^{1^+})$. Then $u_{\varepsilon_i} \geq 0$ on $(0,T)\times\Omega_\varepsilon^p$ for all i.*

Proof. The proof follows exactly as the one for lemma 3.2 given in [Krä08]. Let $\Omega_{\varepsilon_i}^{p^-}(t)$ be the support of $u_{\varepsilon_i}^-(t)$. We multiply the *i-th* PDE of (4.1.8) by $-u_{\varepsilon_i}^-(t)$ and integrate over $\Omega_{\varepsilon_i}^{p^-}(t)$. The rest follows by Gronwall's inequality. ♦

Now we show the existence of a global weak solution of $(P_\varepsilon^{1^+})$. The basic ingredients are a Lyapunov functional, Schaefer's fixed point theorem (see appendix theorem B.1) and a result from [PS01] (cf. theorem 3.3.1). For technical reasons, we add an extra term on both sides of $(P_\varepsilon^{1^+})$, i.e., for a constant $\kappa > 0$ we have

$$\boxed{\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \nabla\cdot D\nabla u_\varepsilon + \kappa u_\varepsilon = S\bar{R}(u_\varepsilon) + \kappa u_\varepsilon \qquad &\text{in} \quad (0,T)\times\Omega_\varepsilon^p, &(4.1.12)\\ u_\varepsilon(0,x) = u_0(x) \qquad &\text{in} \quad \Omega_\varepsilon^p, &(4.1.13)\\ -D\nabla u_\varepsilon\cdot\vec{n} = 0 \qquad &\text{on} \quad (0,T)\times\partial\Omega, &(4.1.14)\\ -D\nabla u_\varepsilon\cdot\vec{n} = 0 \qquad &\text{on} \quad (0,T)\times\Gamma_\varepsilon. &(4.1.15) \end{aligned}}$$

We denote the problem (4.1.12)-(4.1.15) by $(P_{\varepsilon_M}^{1^+})$. We see that a solution of $(P_{\varepsilon_M}^{1^+})$ is also a solution of $(P_\varepsilon^{1^+})$. We prove the global existence of a weak solution of $(P_{\varepsilon_M}^{1^+})$.

4.1.1.1 Schaefer's Fixed Point Operator

Let us define a fixed point operator $Z_1 : \mathcal{F}_\varepsilon^u \to \mathcal{F}_\varepsilon^u$ via

$$Z_1(v_\varepsilon) = u_\varepsilon, \tag{4.1.16}$$

where u_ε is the solution of the linear problem

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D\nabla u_\varepsilon + \kappa u_\varepsilon = S\bar{R}(v_\varepsilon) + \kappa v_\varepsilon \quad \text{in} \quad (0,T)\times\Omega_\varepsilon^p, \tag{4.1.17}$$

$$u_\varepsilon(0,x) = u_0(x) \quad \text{in} \quad \Omega_\varepsilon^p, \tag{4.1.18}$$

$$-D\nabla u_{\varepsilon_i}\cdot\vec{n} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{4.1.19}$$

$$-D\nabla u_{\varepsilon_i}\cdot\vec{n} = 0 \quad \text{on} \quad (0,T)\times\Gamma_\varepsilon. \tag{4.1.20}$$

for $i = 1,2,...,I$.

Remark 4.1.1.1.1. The reformulation of (4.1.17)-(4.1.20) is given by

$$\frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon = f(v_\varepsilon),$$
$$u_\varepsilon(0,x) = u_0(x),$$

where $f(v_\varepsilon) = S\bar{R}(v_\varepsilon) + \kappa v_\varepsilon$ and the operator $A : H^{1,p}(\Omega_\varepsilon^p)^I \to [H^{1,q}(\Omega_\varepsilon^p)^*]^I$ is defined as $Au_\varepsilon := (A_1u_{\varepsilon_1}, A_2u_{\varepsilon_2}, ..., A_Iu_{\varepsilon_I})$ such that for $1 \le i \le I$,

$$\langle A_iu_{\varepsilon_i}, w_{\varepsilon_i}\rangle := \int_{\Omega_\varepsilon^p} D\nabla u_{\varepsilon_i}(x)\cdot\nabla w_{\varepsilon_i}(x)\,dx + \kappa\int_{\Omega_\varepsilon^p} u_{\varepsilon_i}(x)w_{\varepsilon_i}(x)\,dx \quad \text{for } u_{\varepsilon_i}\in H^{1,p}(\Omega_\varepsilon^p) \text{ and } w_{\varepsilon_i}\in H^{1,q}(\Omega_\varepsilon^p), \tag{4.1.21}$$

where $\kappa > 0$. Let us call this reformulated problem as (AP). The assumption (4.1.3) guarantees $u_0 \in \mathcal{X}_{p_\varepsilon}^u$. By theorem 3.4.3.3: Since $v_\varepsilon \in \mathcal{F}_\varepsilon^u$, $v_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon^p)^I$. This shows that $f(v_\varepsilon) = SR(v_\varepsilon) + \kappa v_\varepsilon \in [L^p((0,T);H^{1,q}(\Omega_\varepsilon^p)^*)]^I$.[12] Moreover section 3.3.1 ensures the maximal regularity of A on $[H^{1,q}(\Omega_\varepsilon^p)^*]^I$.[13] Therefore theorem 3.3.1 gives the existence of a unique solution $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ of the problem (AP). Thus the operator Z_1 is well-defined.

Remark 4.1.1.1.2. Every fixed point of Z_1 is a solution of the problem $(P^{1^+}_{\varepsilon_M})$.

In order to use Schaefer's fixed point theorem, we need to verify the following conditions:

(i) The operator Z_1 is continuous and compact.

(ii) The set $\{u_\varepsilon \in \mathcal{F}_\varepsilon^u | \exists\lambda\in[0,1] : u_\varepsilon = \lambda Z_1(u_\varepsilon)\}$ is bounded, i.e., there exists a constant $C_{10} > 0$ such that any arbitrary solution $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ of the equation

$$u_\varepsilon = \lambda Z_1(u_\varepsilon) \tag{4.1.22}$$

satisfies

$$|||u_\varepsilon|||_{\mathcal{F}_\varepsilon^u} \le C_{10}, \tag{4.1.23}$$

where C_{10} is independent of λ, ε, u_ε and t. Equations (4.1.17)-(4.1.20) and (4.1.22) imply

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla\cdot D\nabla u_\varepsilon + \kappa u_\varepsilon = \lambda S\bar{R}(u_\varepsilon) + \lambda\kappa u_\varepsilon \quad \text{in} \quad (0,T)\times\Omega_\varepsilon^p, \tag{4.1.24}$$

$$u_\varepsilon(0,x) = \lambda u_0(x) \quad \text{in} \quad \Omega_\varepsilon^p, \tag{4.1.25}$$

$$-D\nabla u_\varepsilon\cdot\vec{n} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{4.1.26}$$

$$-D\nabla u_\varepsilon\cdot\vec{n} = 0 \quad \text{on} \quad (0,T)\times\Gamma_\varepsilon. \tag{4.1.27}$$

We denote the problem (4.1.24)-(4.1.27) as $(P^{1^+}_{\varepsilon_{\lambda_M}})$.

[12] We have used $L^p(\Omega_\varepsilon^p) \hookrightarrow H^{1,q}(\Omega_\varepsilon^p)^*$.

[13] The operator A is said to have maximal regularity on $[H^{1,q}(\Omega_\varepsilon^p)^*]^I$ if each A_i has maximal regularity on $H^{1,q}(\Omega_\varepsilon^p)^*$.

4.1.1.2 Introduction of the Lyapunov Functions

Let $\mu^0 \in \mathbb{R}^I$ be a solution of the linear system

$$S^T \mu^0 = -\log K, \tag{4.1.28}$$

where $K \in \mathbb{R}^J$ is the vector of equilibrium constants $K_j = \frac{k_j^f}{k_j^b}$ related to the J kinetic reactions. Due to assumption (4.1.4), the system (4.1.24) has a solution μ^0. As in [Krä08], we define the following functions:

Let $g_i : \mathbb{R}_0^+ \to \mathbb{R}$ and $g : \mathbb{R}_0^{+I} \to \mathbb{R}$ be defined as[14]

$$g_i(u_{\varepsilon_i}) = (\mu_i^0 - 1 + \log u_{\varepsilon_i})u_{\varepsilon_i} + e^{(1-\mu_i^0)} \text{ for each } i = 1, 2, ..., I$$

and

$$g(u_\varepsilon) = \sum_{i=1}^{I} g_i(u_{\varepsilon_i}).$$

Also for $r \in \mathbb{N}$, we define $f_r : \mathbb{R}_0^{+I} \to \mathbb{R}$ and $F_r : L_+^\infty(\Omega_\varepsilon^p)^I \to \mathbb{R}$ as

$$f_r(u_\varepsilon) = [g(u_\varepsilon)]^r$$

and

$$F_r(u_\varepsilon) = \int_{\Omega_\varepsilon^p} f_r(u_\varepsilon(x))\, dx.$$

Proposition 4.1.1.2.1. *For all $i = 1, 2, ..., I$ and $\varepsilon > 0$,*

$$g(u_\varepsilon) \geq g_i(u_{\varepsilon_i}) \geq u_{\varepsilon_i} \tag{4.1.29}$$

and

$$F_r(u_\varepsilon) \geq ||u_{\varepsilon_i}||^r_{L^r(\Omega_\varepsilon^p)}. \tag{4.1.30}$$

Proof. The inequality (4.1.29) is straightforward. For (4.1.30) see that

$$F_r(u_\varepsilon) = \int_{\Omega_\varepsilon^p} f_r(u_\varepsilon(x))\, dx = \int_{\Omega_\varepsilon^p} [g(u_\varepsilon(x))]^r\, dx \geq \int_{\Omega_\varepsilon^p} |u_{\varepsilon_i}(x)|^r\, dx.$$

♦

Proposition 4.1.1.2.2. *Let $\alpha > 0$. There exist constants $C_{11}, C_{12}, C_{13} > 0$ depending on α and μ_i but independent of ε and u_{ε_i} such that*

$$g_i(u_{\varepsilon_i}) \leq C_{11}(1 + u_{\varepsilon_i}^{1+\alpha}) \quad \textit{for all } i, \tag{4.1.31}$$

$$g(u_\varepsilon) \leq C_{12}(1 + |u_\varepsilon|_I^{1+\alpha}) \tag{4.1.32}$$

and

$$f_r(u_\varepsilon) \leq C_{13}(1 + |u_\varepsilon|_I^{r(1+\alpha)}) \tag{4.1.33}$$

[14] Here we have considered the natural logarithm, i.e. $\log_e u_{\varepsilon_i}$.

Proof. The proof follows from the definitions of g_i, g and f_r. ♦

From (4.1.30) it is clear that the L^r - norm of u_{ε_i} will be finite if we can obtain an upper bound of $F_r(u_\varepsilon)$. This is the main concern of the following theorem:

Theorem 4.1.1.2.3. *Let $r \in \mathbb{N}\,(r \geq 2)$, $0 \leq t \leq T$ and $0 \leq \lambda \leq 1$. Further assume that $u_\varepsilon \in \mathcal{F}^u_\varepsilon$ is a solution of $(P^{1+}_{\varepsilon_{\lambda_M}})$. Then the following inequality holds good:*

$$F_r(u_\varepsilon(t)) \leq e^{Ir\kappa(e(e-1))^{-1}t} F_r(u_\varepsilon(0)) \quad \textit{for a.e. } t \textit{ and for all } r. \tag{4.1.34}$$

To prove this theorem, we need the following lemmas as basic ingredients. For $p > n+1$ and $\zeta \in [H^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);H^{2,p}(\Omega))]^I$, these lemmas have been proved in [Krä08] but they can be adapted for the functions in $\mathcal{F}^u_\varepsilon$ with $p > n+2$.

Lemma 4.1.1.2.4. *Let $p > n+2$. The map $F_r : L^\infty_+(\Omega^p_\varepsilon)^I \to \mathbb{R}$ is continuous.*

Proof. The proof is analogous to the proof of the lemma 3.4 in [Krä08]. ♦

Let us consider the derivative (in the classical sense) of $f_r : \mathbb{R}^{+I}_0 \to \mathbb{R}^I$ which is given as

$$\begin{aligned}
\partial f_r(v_\varepsilon) &= \nabla_{v_\varepsilon} f_r(v_\varepsilon) \\
&= r[g(v_\varepsilon)]^{r-1} \nabla_{v_\varepsilon} g(v_\varepsilon) \\
&= r f_{r-1}(v_\varepsilon)\left(\mu^0 + \log v_\varepsilon\right).
\end{aligned}$$

We see that $\partial f_r(v_\varepsilon)$ is undefined for $v_\varepsilon = 0$ whereas $f_{r-1}(v_\varepsilon)$ is defined for all $v_\varepsilon \geq 0$. Since we only know the nonnegativity of v_ε, for any $\delta > 0$, we define

$$v_{\varepsilon_\delta} := v_\varepsilon + \delta. \tag{4.1.35}$$

Clearly, $v_{\varepsilon_\delta} \geq \delta > 0$ and $v_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$. From here on we work with the function v_{ε_δ} unless stated otherwise. We aim to prove that for $v_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$,

$$\partial f_r(v_{\varepsilon_\delta}) \in L^q((0,T);H^{1,q}(\Omega^p_\varepsilon))^I. \tag{4.1.36}$$

To prove (4.1.36), our point of departure is the following lemma which deals with the continuity of ∂f_r.

Lemma 4.1.1.2.5. *Let $p > n+2$ and $\delta > 0$, then the map*

$$v_{\varepsilon_\delta} \mapsto \partial f_r(v_{\varepsilon_\delta}), \text{ i.e., } \partial f_r : \mathcal{F}^u_\varepsilon \to L^\infty((0,T)\times\Omega^p_\varepsilon)^I$$

is continuous.

Proof. Let $v_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$. For $p > n+2$, from theorem 3.4.3.3 it follows that $v_{\varepsilon_\delta} \in [L^\infty((0,T)\times\Omega^p_\varepsilon)]^I$. The rest follows as in lemma 3.6 in [Krä08]. ♦

Lemma 4.1.1.2.6. *(Derivative of the vector function $x \longmapsto \partial f_r(v_{\varepsilon_\delta}(t,x))$ w.r.t. $x \in \Omega^p_\varepsilon$) Let $p > n+2$, $r \in \mathbb{N}\,(r \geq 2)$ and $v_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$. We define the mapping $w(v_{\varepsilon_\delta}) : (0,T)\times\Omega^p_\varepsilon \to \mathbb{R}^{I\times n}$ by*

$$w(v_{\varepsilon_\delta})(t,x) := \{r(r-1)f_{r-2}(v_{\varepsilon_\delta})M_\mu(v_{\varepsilon_\delta}) + r f_{r-1}(v_{\varepsilon_\delta})\Lambda_{\frac{1}{v_{\varepsilon_\delta}}}\}\nabla_x v_{\varepsilon_\delta}(t,x), \tag{4.1.37}$$

where $M_\mu(v_{\varepsilon_\delta})$ is the $I\times I$-th order symmetric matrix with entries $(\mu^0_i + \log v_{\varepsilon_{\delta_i}})(\mu^0_j + \log v_{\varepsilon_{\delta_j}})$ and $\Lambda_{\frac{1}{v_{\varepsilon_\delta}}}$ is the $I\times I$-th order diagonal matrix with entries $\frac{1}{v_{\varepsilon_{\delta_i}}}$. Then

$$\nabla_x(\partial f_r(v_{\varepsilon_\delta})) = w(v_{\varepsilon_\delta}) \in L^q((0,T);L^q(\Omega^p_\varepsilon))^{I\times n}, \tag{4.1.38}$$

i.e.,

$$\partial f_r(v_{\varepsilon_\delta}) \in L^q((0,T);H^{1,q}(\Omega^p_\varepsilon))^I. \tag{4.1.39}$$

Proof. Let $v_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$. For $p > n+2$, theorem 3.4.3.3 implies $v_{\varepsilon_\delta} \in L^\infty((0,T)\times\Omega^p_\varepsilon)^I$. Since $v_{\varepsilon_\delta} \geq \delta$, from the definitions of $f_r(v_{\varepsilon_\delta})$, $M_\mu(v_{\varepsilon_\delta})$ and $\Lambda_{\frac{1}{v_{\varepsilon_\delta}}}$, we have

$$r(r-1)f_{r-2}(v_{\varepsilon_\delta})M_\mu(v_{\varepsilon_\delta}) + rf_{r-1}(v_{\varepsilon_\delta})\Lambda_{\frac{1}{v_{\varepsilon_\delta}}} \in L^\infty((0,T)\times\Omega^p_\varepsilon)^{I\times I}. \tag{4.1.40}$$

Also note that for $p > n+2$ and $v_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$, $\nabla_x v_{\varepsilon_\delta} \in L^q((0,T);L^q(\Omega^p_\varepsilon))^{I\times n}$. Therefore $w(v_{\varepsilon_\delta}) \in L^q((0,T);L^q(\Omega^p_\varepsilon))^{I\times n}$. Next we prove that $\nabla_x(\partial f_r(v_{\varepsilon_\delta})) = w(v_{\varepsilon_\delta})$. This follows from the density of $C^\infty((0,T)\times\Omega^p_\varepsilon)^I$ in $\mathcal{F}^u_\varepsilon$ (for details confer lemma 3.6 in [Krä08]). ♦

Lemma 4.1.1.2.7. *Let $u_\varepsilon \in \mathcal{F}^u_\varepsilon$ be the solution of the problem $(P^{1^+}_{\varepsilon_{\lambda_M}})$ and $\delta > 0$ be such that $u_{\varepsilon_\delta} := u_\varepsilon + \delta$. Then we have the following inequality*

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u_\varepsilon}{\partial\theta}, \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta \\
&\qquad \leq Ir\kappa(e(e-1))^{-1}\int_0^t F_r(u_{\varepsilon_\delta}(\theta))\,d\theta + h(t,u_{\varepsilon_\delta},\delta) + l(t,u_{\varepsilon_\delta},\delta) \ \ \textit{for a.e. } t,
\end{aligned} \tag{4.1.41}$$

where $h(t,\delta,u_{\varepsilon_\delta})$ and $l(t,\delta,u_{\varepsilon_\delta}) \longrightarrow 0$ as $\delta \longrightarrow 0$ for a.e. t.

Proof. From lemma 4.1.1.2, the nonnegativity of the solution of $(P^{1^+}_\varepsilon)$ implies that the solution of $(P^{1^+}_{\varepsilon_\lambda})$ is also nonnegative, hence

$$u_{\varepsilon_\delta} = u_\varepsilon + \delta \geq \delta.$$

Clearly, $u_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$. By lemma 4.1.1.2.6, $\partial f_r(u_{\varepsilon_\delta}) \in L^q((0,T);H^{1,q}(\Omega^p_\varepsilon))^I$. Using $\partial f_r(u_{\varepsilon_\delta})$ as the test function for the weak formulation of $(P^{1^+}_{\varepsilon_\lambda})$, we obtain

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u_\varepsilon}{\partial\theta}, \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta \\
&\quad - \int_0^t \langle \nabla D\nabla u_\varepsilon, \partial f_r(u_{\varepsilon_\delta})\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I}\, d\theta + \kappa\int_0^t\int_{\Omega^p_\varepsilon}\langle u_\varepsilon, \partial f_r(u_{\varepsilon_\delta})\rangle_I\,dx\,d\theta \\
&\qquad = \lambda\int_0^t \left\langle S\bar{R}(u_\varepsilon), \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta + \lambda\kappa\int_0^t\int_{\Omega^p_\varepsilon}\langle u_\varepsilon, \partial f_r(u_{\varepsilon_\delta})\rangle_I\,dx\,d\theta,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u_\varepsilon}{\partial\theta}, \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta \\
&\quad = -\int_0^t \langle D\nabla u_{\varepsilon_\delta}, \nabla_x(\partial f_r(u_{\varepsilon_\delta}))\rangle_{[L^p(\Omega^p_\varepsilon)]^{I\times n}\times[L^q(\Omega^p_\varepsilon)]^{I\times n}}\, d\theta \\
&\qquad + \lambda\int_0^t \left\langle S\bar{R}(u_\varepsilon), \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta - (1-\lambda)\kappa\int_0^t\int_{\Omega^p_\varepsilon}\langle u_\varepsilon, \partial f_r(u_{\varepsilon_\delta})\rangle_I\,dx\,d\theta,
\end{aligned}$$

i.e.,

$$\int_0^t \left\langle \frac{\partial u_\varepsilon}{\partial\theta}, \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta \quad =: I^{(t)}_{diff} + I^{(t)}_{reac} + I^{(t)}_{Ex} \qquad \text{for a.e. } t, \tag{4.1.42}$$

where

$$\begin{aligned}
I^{(t)}_{diff} &:= -\sum_{k=1}^n \int_0^t\int_{\Omega^p_\varepsilon}\left\langle D\frac{\partial}{\partial x_k}u_{\varepsilon_\delta}, \frac{\partial}{\partial x_k}(\partial f_r(u_{\varepsilon_\delta}))\right\rangle_I dx\,d\theta \\
I^{(t)}_{reac} &:= \lambda\int_0^t \left\langle S\bar{R}(u_\varepsilon), \partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)^*]^I\times[H^{1,q}(\Omega^p_\varepsilon)]^I} d\theta
\end{aligned}$$

and

$$I_{Ex}^{(t)} := -(1-\lambda)\kappa \int_0^t \int_{\Omega_\varepsilon^p} \langle u_\varepsilon, \partial f_r(u_{\varepsilon\delta})\rangle_I \, dx \, d\theta.$$

Now we simplify the terms $I_{diff}^{(t)}$, $I_{reac}(t)$ and $I_{Ex}(t)$ one by one. [15]

$$\begin{aligned} I_{reac}^{(t)} &= \lambda \int_0^t \int_{\Omega_\varepsilon^p} \left\langle S\bar{R}(u_\varepsilon), \partial f_r(u_{\varepsilon\delta})\right\rangle_I dx \, d\theta \\ &= \lambda \int_0^t \int_{\Omega_\varepsilon^p} \left\langle r f_{r-1}(u_{\varepsilon\delta})\left(\mu^0 + \log u_{\varepsilon\delta}\right), S\bar{R}(u_\varepsilon)\right\rangle_I dx \, d\theta \\ &= \lambda r \int_0^t \int_{\Omega_\varepsilon^p} f_{r-1}(u_{\varepsilon\delta})\left\langle \mu^0 + \log u_{\varepsilon\delta}, S\bar{R}(u_\varepsilon)\right\rangle_I dx \, d\theta \quad \text{for a.e. } t. \end{aligned} \tag{4.1.43}$$

Following the steps of lemma 3.7 in [Krä08], we can estimate the integral on the r.h.s. of (4.1.43), i.e.,

$$\begin{aligned} I_{reac}^{(t)} \quad &\leq \lambda r C \sum_{i=1}^{I} \left(\int_0^t \int_{\Omega_\varepsilon^p} \left(\delta|\mu_i^0| + T|\Omega_\varepsilon^p||\delta|\log\delta|\right) dx \, d\theta \right. \\ &\qquad \left. + \delta \int_0^t \int_{\Omega_\varepsilon^p} (u_{\varepsilon_i} + \delta) \, dx \, d\theta \right) =: h(t, u_{\varepsilon\delta}, \delta) \text{ for a.e. } t, \end{aligned}$$

where C is independent of λ and $u_{\varepsilon\delta}$, and all the other factors of $h(t, u_{\varepsilon\delta}, \delta)$ are bounded and tending to zero as $\delta \longrightarrow 0$ for a.e. t, i.e.,

$$I_{reac}^{(t)} \leq h(t, u_{\varepsilon\delta}, \delta) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0 \quad \text{for a.e. } t. \tag{4.1.44}$$

From lemma 5.8 in [Krä08] we get

$$\begin{aligned} I_{diff}^{(t)} &= -\sum_{k=1}^{n} \int_0^t \int_{\Omega_\varepsilon^p} \left\langle D \frac{\partial}{\partial x_k} u_{\varepsilon\delta}, \frac{\partial}{\partial x_k}\left(\partial f_r(u_{\varepsilon\delta})\right)\right\rangle_I dx \, d\theta \\ &= -r(r-1)D \int_0^t \int_{\Omega_\varepsilon^p} f_{r-2}(u_{\varepsilon\delta}) \sum_{k=1}^{n} \left\langle \mu^0 + \log u_{\varepsilon\delta}, \partial_{x_k} u_{\varepsilon\delta}\right\rangle_I^2 dx \, d\theta \\ &\quad - rD \int_0^t \int_{\Omega_\varepsilon^p} f_{r-1}(u_{\varepsilon\delta}) \sum_{i=1}^{I} \sum_{k=1}^{n} \frac{1}{u_{\varepsilon\delta_i}} \left(\frac{\partial u_{\varepsilon\delta_i}}{\partial x_k}\right)^2 dx \, d\theta \quad \text{for a.e. } t. \end{aligned} \tag{4.1.45}$$

Both the terms of (4.1.45) are nonpositive, hence

$$I_{diff}^{(t)} \leq 0 \quad \text{for a.e. } t. \tag{4.1.46}$$

[15] We have $p > n+2$. Then $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ implies that $u_\varepsilon \in L^\infty((0,T) \times \Omega_\varepsilon^p)^I$. This gives $SR(u_\varepsilon) \in L^p((0,T); L^p(\Omega_\varepsilon^p))^I \hookrightarrow L^p((0,T); H^{1,q}(\Omega_\varepsilon^p)^*)^I$. Recall the definition (3.1.3) for the continuous embedding $L^p(\Omega_\varepsilon^p) \hookrightarrow H^{1,q}(\Omega_\varepsilon^p)^*$ as

$$\langle f, \zeta\rangle_{H^{1,q}(\Omega_\varepsilon^p)^* \times H^{1,q}(\Omega_\varepsilon^p)} = \langle f, \zeta\rangle_{L^p(\Omega_\varepsilon^p) \times L^q(\Omega_\varepsilon^p)}, \text{ for } f \in L^p(\Omega_\varepsilon^p) \text{ and } \zeta \in H^{1,q}(\Omega_\varepsilon^p).$$

$$
\begin{aligned}
I_{Ex}^{(t)} &= -\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} u_{\varepsilon_i}\partial f_r(u_{\varepsilon_\delta})_i\,dx\,d\theta \\
&= \kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} r(\delta-u_{\varepsilon_{\delta_i}})f_{r-1}(u_{\varepsilon_\delta})(\mu_i^0+\log u_{\varepsilon_{\delta_i}})\,dx\,d\theta \quad \text{since } u_{\varepsilon_{\delta_i}}=u_{\varepsilon_i}+\delta \\
&= \delta\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} r(\mu_i^0+\log u_{\varepsilon_{\delta_i}})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta \\
&\qquad + r\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} -u_{\varepsilon_{\delta_i}}(\mu_i^0+\log u_{\varepsilon_{\delta_i}})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta
\end{aligned}
\tag{4.1.47}
$$

It can be shown that

$$
-u_{\varepsilon_{\delta_i}}(\mu_i^0+\log u_{\varepsilon_{\delta_i}})\le e^{-(1+\mu_i^0)} \qquad \forall i. \tag{4.1.48}
$$

We have $\log u_{\varepsilon_{\delta_i}}\le u_{\varepsilon_{\delta_i}}\le g_i(u_{\varepsilon_{\delta_i}})$ and $g_i(u_{\varepsilon_{\delta_i}})\ge(e-1)e^{-\mu_i^0}$. Choosing a constant $C=\max\limits_{1\le i\le I}\left(1+|\mu_i^0|\,e^{-\mu_i^0}(e-1)\right)$, we obtain

$$
\mu_i^0+\log u_{\varepsilon_{\delta_i}}\le \mu_i^0+g_i(u_{\varepsilon_{\delta_i}})\le\left|\mu_i^0\right|+g_i(u_{\varepsilon_{\delta_i}})\le C\,g_i(u_{\varepsilon_{\delta_i}}) \tag{4.1.49}
$$

Combining (4.1.47), (4.1.48) and (4.1.49), we get

$$
\begin{aligned}
I_{Ex}^{(t)} &\le (1-\lambda)\left[r\delta\kappa\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} Cg_i(u_{\varepsilon_{\delta_i}})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta+\kappa\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} re^{-(1+\mu_i^0)}f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta\right] \\
&\le r\delta\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} Cg_i(u_{\varepsilon_{\delta_i}})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta \\
&\qquad+\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} r(e(e-1))^{-1}g_i(u_{\varepsilon_{\delta_i}})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta \\
&\le r\delta\kappa(1-\lambda)C\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} g(u_{\varepsilon_\delta})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta \\
&\qquad+\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_{\Omega_\varepsilon^p} r(e(e-1))^{-1}g(u_{\varepsilon_\delta})f_{r-1}(u_{\varepsilon_\delta})\,dx\,d\theta \quad \text{since } g_i(u_{\varepsilon_{\delta_i}})\le g(u_{\varepsilon_\delta}) \\
&\le Ir\kappa\delta\,C\int_0^t\int_{\Omega_\varepsilon^p} f_r(u_{\varepsilon_\delta})\,dx\,d\theta+Ir\kappa(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(u_{\varepsilon_\delta})\,dx\,d\theta \quad \text{since } 0\le\lambda\le 1 \\
&\qquad\qquad \text{and } f_r=f_{r-1}g \quad \text{for a.e. } t.
\end{aligned}
\tag{4.1.50}
$$

As $\delta\to 0$, $f_r(u_{\varepsilon_\delta})$ is bounded in $L^1((0,T)\times\Omega)$. Therefore for a.e. t the first term in (4.1.50) tends to zero as $\delta\to 0$. Denote the first term by $l(t,u_{\varepsilon_\delta},\delta)$, then

$$
I_{Ex}^{(t)}\le l(t,u_{\varepsilon_\delta},\delta)+Irk(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(u_{\varepsilon_\delta})\,dx\,d\theta \quad \text{for a.e. } t. \tag{4.1.51}
$$

Therefore combining (4.1.42), (4.1.44), (4.1.46) and (4.1.51) we obtain

$$
\begin{aligned}
&\int_0^t\left\langle\frac{\partial u_\varepsilon}{\partial\theta},\partial f_r(u_{\varepsilon_\delta})\right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^I\times[H^{1,q}(\Omega_\varepsilon^p)]^I} d\theta \\
&= I_{diff}^{(t)}+I_{reac}^{(t)}+I_{Ex}^{(t)} \\
&\le 0+h(t,u_{\varepsilon_\delta},\delta)+l(t,u_{\varepsilon_\delta},\delta)+Ir\kappa(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(u_{\varepsilon_\delta})\,dx\,d\theta \\
&\le h(t,u_{\varepsilon_\delta},\delta)+l(t,u_{\varepsilon_\delta},\delta)+Ir\kappa(e(e-1))^{-1}\int_0^t F_r(u_{\varepsilon_\delta})\,d\theta \quad \text{for a.e. } t,
\end{aligned}
$$

where $h(t,\delta,u_{\varepsilon_\delta})$ and $l(t,u_{\varepsilon_\delta},\delta)\longrightarrow 0$ as $\delta\longrightarrow 0$ for a.e. t. ♦

Proof of theorem 4.1.1.2.3. Let $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ be the solution of $(P^{1^+}_{\varepsilon_{\lambda_M}})$. Due to lemma 4.1.1.2 we know that $u_\varepsilon \geq 0$. For any fixed $\delta > 0$, let

$$u_{\varepsilon_\delta} := u_\varepsilon + \delta.$$

Let us choose a positive constant η and a smooth function $\bar{u}_{\varepsilon_\delta} \in C^\infty([0,T]\times\bar{\Omega}_\varepsilon^p)^I$ sufficiently close to u_{ε_δ} such that

$$\bar{u}_{\varepsilon_\delta} \geq \frac{\delta}{2}, \tag{4.1.52}$$

$$||\partial_t \bar{u}_{\varepsilon_\delta} - \partial_t u_{\varepsilon_\delta}||_{L^p((0,T);H^{1,q}(\Omega_\varepsilon^p)^*)^I} \leq \eta, \tag{4.1.53}$$

$$|[F_r(u_{\varepsilon_\delta}(t)) - F_r(u_{\varepsilon_\delta}(0))] - [F_r(\bar{u}_{\varepsilon_\delta}(t)) - F_r(\bar{u}_{\varepsilon_\delta}(0))]| \leq \delta, \tag{4.1.54}$$

$$||\partial f_r(u_{\varepsilon_\delta}) - \partial f_r(\bar{u}_{\varepsilon_\delta})||_{L^\infty((0,T)\times\Omega_\varepsilon^p)^I} \leq \eta, \tag{4.1.55}$$

and

$$\eta\,||\partial f_r(u_{\varepsilon_\delta})||_{L^q((0,T);H^{1,q}(\Omega_\varepsilon^p))^I} + \eta\,||\partial_t \bar{u}_{\varepsilon_\delta}||_{L^1((0,T)\times\Omega_\varepsilon^p)^I} \leq \delta. \tag{4.1.56}$$

Then

$$\begin{aligned}
&\left|\int_0^t \langle \partial f_r(u_{\varepsilon_\delta}), \partial_\theta u_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta - \int_0^t \langle \partial f_r(\bar{u}_{\varepsilon_\delta}), \partial_\theta \bar{u}_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta\right| \\
&= \left|\int_0^t \langle \partial f_r(u_{\varepsilon_\delta}) - \partial f_r(\bar{u}_{\varepsilon_\delta}), \partial_\theta \bar{u}_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta\right. \\
&\qquad\left. + \int_0^t \langle \partial_\theta u_{\varepsilon_\delta} - \partial_\theta \bar{u}_{\varepsilon_\delta}, \partial f_r(u_{\varepsilon_\delta})\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta\right| \\
&\leq \sum_{i=1}^I \int_0^T \left|\left\langle \partial f_r(u_{\varepsilon_\delta})_i - \partial f_r(\bar{u}_{\varepsilon_\delta})_i, \partial_\theta \bar{u}_{\varepsilon_{\delta_i}}\right\rangle_{H^{1,q}(\Omega_\varepsilon^p)\times H^{1,q}(\Omega_\varepsilon^p)^*}\right| d\theta \\
&\qquad + \sum_{i=1}^I \int_0^T \left|\left\langle \partial_\theta u_{\varepsilon_{\delta_i}} - \partial_\theta \bar{u}_{\varepsilon_{\delta_i}}, \partial f_r(u_{\varepsilon_\delta})_i\right\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}\right| d\theta \\
&\leq \sum_{i=1}^I \int_0^T \left|\left\langle \partial f_r(u_{\varepsilon_\delta})_i - \partial f_r(\bar{u}_{\varepsilon_\delta})_i, \partial_\theta \bar{u}_{\varepsilon_{\delta_i}}\right\rangle_{L^q(\Omega_\varepsilon^p)\times L^p(\Omega_\varepsilon^p)}\right| d\theta \\
&\qquad + \sum_{i=1}^I \int_0^T \left|\left\langle \partial_\theta u_{\varepsilon_{\delta_i}} - \partial_\theta \bar{u}_{\varepsilon_{\delta_i}}, \partial f_r(u_{\varepsilon_\delta})_i\right\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}\right| d\theta \\
&\leq \sum_{i=1}^I \Big[||\partial f_r(u_{\varepsilon_\delta})_i - \partial f_r(\bar{u}_{\varepsilon_\delta})_i||_{L^\infty((0,T)\times\Omega_\varepsilon^p)} \left|\left|\partial_t \bar{u}_{\varepsilon_{\delta_i}}\right|\right|_{L^1((0,T)\times\Omega_\varepsilon^p)} \\
&\qquad + \left|\left|\partial_t u_{\varepsilon_{\delta_i}} - \partial_t \bar{u}_{\varepsilon_{\delta_i}}\right|\right|_{L^p((0,T);H^{1,q}(\Omega_\varepsilon^p)^*)} ||\partial f_r(u_{\varepsilon_\delta})_i||_{L^q((0,T);H^{1,q}(\Omega_\varepsilon^p))}\Big] \\
&\leq \sum_{i=1}^I \Big[\eta \left|\left|\partial_t \bar{u}_{\varepsilon_{\delta_i}}\right|\right|_{L^1((0,T)\times\Omega_\varepsilon^p)} + \eta\, ||\partial f_r(u_{\varepsilon_\delta})_i||_{L^q((0,T);H^{1,q}(\Omega_\varepsilon^p))}\Big] \\
&\leq \sum_{i=1}^I \delta \;=\; \delta I \qquad \text{by (4.1.56).}
\end{aligned} \tag{4.1.57}$$

For the smooth function $\bar{u}_{\varepsilon_\delta}$, we have

$$
\begin{aligned}
F_r(\bar{u}_{\varepsilon_\delta}(t)) - F_r(\bar{u}_{\varepsilon_\delta}(0)) &= \int_0^t \frac{d}{d\theta}\left(F_r(\bar{u}_{\varepsilon_\delta}(\theta))\right) d\theta \\
&= \int_0^t \frac{d}{d\theta} \int_{\Omega_\varepsilon^p} f_r(\bar{u}_{\varepsilon_\delta})\, dx\, d\theta \\
&= \int_0^t \int_{\Omega_\varepsilon^p} \frac{\partial}{\partial\theta} f_r(\bar{u}_{\varepsilon_\delta})\, dx\, d\theta \\
&= \sum_{i=1}^I \int_0^t \int_{\Omega_\varepsilon^p} \partial f_r(\bar{u}_{\varepsilon_\delta})_i \frac{\partial \bar{u}_{\varepsilon_{\delta_i}}}{\partial\theta}\, dx\, d\theta \\
&= \sum_{i=1}^I \int_0^t \langle \partial f_r(\bar{u}_{\varepsilon_\delta})_i, \frac{\partial \bar{u}_{\varepsilon_{\delta_i}}}{\partial\theta} \rangle_{H^{1,q}(\Omega_\varepsilon^p)\times H^{1,q}(\Omega_\varepsilon^p)^*}\, d\theta \\
&= \int_0^t \left\langle \partial f_r(\bar{u}_{\varepsilon_\delta}), \frac{\partial \bar{u}_{\varepsilon_\delta}}{\partial\theta} \right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I} d\theta. \qquad (4.1.58)
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\left| F_r(u_{\varepsilon_\delta}(t)) - F_r(u_{\varepsilon_\delta}(0)) - \int_0^t \langle \partial f_r(u_{\varepsilon_\delta}), \partial_\theta u_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta \right| \\
&\le \left| [F_r(u_{\varepsilon_\delta}(t)) - F_r(u_{\varepsilon_\delta}(0))] - [F_r(\bar{u}_{\varepsilon_\delta}(t)) - F_r(\bar{u}_{\varepsilon_\delta}(0))] \right| \\
&\qquad + \left| \int_0^t \langle \partial f_r(\bar{u}_{\varepsilon_\delta}), \partial_\theta \bar{u}_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta \right. \\
&\qquad\qquad \left. - \int_0^t \langle \partial f_r(u_{\varepsilon_\delta}), \partial_\theta u_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta \right| \quad \text{by (4.1.58)} \\
&\le \delta + \delta I \quad \text{by} \quad (4.1.54) \quad \text{and} \quad (4.1.57) \\
&\le (I+1)\delta.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&|F_r(u_{\varepsilon_\delta}(t)) - F_r(u_{\varepsilon_\delta}(0))| \\
&\le (I+1)\delta + \int_0^t \langle \partial f_r(u_{\varepsilon_\delta}), \partial_\theta u_{\varepsilon_\delta}\rangle_{[H^{1,q}(\Omega_\varepsilon^p)]^I\times[H^{1,q}(\Omega_\varepsilon^p)^*]^I}\, d\theta \\
&\le (I+1)\delta + h(t,u_{\varepsilon_\delta},\delta) + l(t,u_{\varepsilon_\delta},\delta) + Ir\kappa(e(e-1))^{-1}\int_0^t F_r(u_{\varepsilon_\delta})\, d\theta \quad \text{by lemma 4.1.1.2.7}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.1.59)
\end{aligned}
$$

where $h(t,u_{\varepsilon_\delta},\delta)$, $l(t,u_{\varepsilon_\delta},\delta) \to 0$ as $\delta\to 0$ for a.e. t. Therefore from the continuity of F_r (4.1.59) reduces to

$$
F_r(u_\varepsilon(t)) \le F_r(u_\varepsilon(0)) + Ir\kappa(e(e-1))^{-1}\int_0^t F_r(u)\, d\theta \text{ for a.e } t.
$$

Gronwall's inequality gives

$$
\boxed{F_r(u_\varepsilon(t)) \le e^{Ir\kappa(e(e-1))^{-1}t} F_r(u_\varepsilon(0)) \quad \text{for all } r \text{ and for a.e. } t.}
$$

This completes the proof. ♦

An immediate consequence of theorem 4.1.1.2.3 is the following corollary which gives the *a-priori* estimates (global in time) of the solution of $(P^{1^+}_{\varepsilon_{\lambda_M}})$. For all $r\in\mathbb{N}$, let us define

$$C_{14} := C_{14}(r) := \left[I \operatorname*{ess\,sup}_{t\in(0,T)} \sup_i \sup_{\varepsilon>0} C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t} |\Omega| \left(1+\left(I^{\frac{1}{2}}||u_0||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right)\right]^{\frac{1}{r}}$$

and $C_{15} := \sup_{\varepsilon>0} \left(1+\left(I^{\frac{1}{2}}\,|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{1+\alpha}\right)$.

Corollary 4.1.1.2.8. *Let $p>n+2$, $r\in\mathbb{N}$ $(2\le r<\infty)$ and $0\le\lambda\le 1$. Suppose that $u_\varepsilon\in\mathcal{F}_\varepsilon^u$ is the solution of the problem $(P^{1^+}_{\varepsilon_{\lambda_M}})$, then the following estimates holds:*

$$\sup_{\varepsilon>0} |||u_\varepsilon(t)|||_{L^r(\Omega_\varepsilon^p)^I} \le C_{14} < \infty \quad \textit{for all } r \textit{ and for a.e. } t, \tag{4.1.60}$$

and

$$\sup_{\varepsilon>0} |||u_\varepsilon(t)|||_{L^\infty(\Omega_\varepsilon^p)^I} \le C_{15} < \infty \quad \textit{for a.e. } t. \tag{4.1.61}$$

Proof. From theorem 3.4.3.3, it follows that for $p>n+2$, $u_0\in L^\infty(\Omega_\varepsilon^p)^I$. For the problem $(P^{1^+}_{\varepsilon_{\lambda_M}})$, $u_\varepsilon(0)=\lambda u_0$ and $\sup_{\varepsilon>0} |||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}<\infty$. Therefore from theorem 4.1.1.2.3, for $0\le t\le T$, we have

$$\begin{aligned}
&F_r(u_\varepsilon(t)) \le e^{Ir\kappa(e(e-1))^{-1}t} F_r(u_\varepsilon(0)) \quad \text{for all } r \text{ and for a.e. } t\\
\Longrightarrow\quad & \int_{\Omega_\varepsilon^p} f_r(u_\varepsilon(t,x))\,dx \le e^{Ir\kappa(e(e-1))^{-1}t} F_r(\lambda u_0) \quad \text{for all } r \text{ and for a.e. } t\\
\Longrightarrow\quad & \int_{\Omega_\varepsilon^p} u_{\varepsilon_i}^r(t,x)\,dx \le e^{Ir\kappa(e(e-1))^{-1}t} \int_{\Omega_\varepsilon^p} f_r(\lambda u_0(x))\,dx \quad \text{for all } r \text{ and for a.e. } t.
\end{aligned} \tag{4.1.62}$$

From proposition 4.1.1.2.2, we have

$$f_r(\lambda u_0) \le C_{13}\left(1+|\lambda u_0|_I^{r(1+\alpha)}\right), \tag{4.1.63}$$

where $\alpha>0$ and C_{13} is independent of ε, δ, λ and u_{ε_i}. Combining (4.1.62) and (4.1.63), we obtain

$$\begin{aligned}
&||u_{\varepsilon_i}(t)||^r_{L^r(\Omega_\varepsilon^p)}\\
&\le C_{13}\, e^{Ir\kappa(e(e-1))^{-1}t} \int_{\Omega_\varepsilon^p} (1+|u_0|_I^{r(1+\alpha)})\,dx, \quad \text{since} \quad 0\le\lambda\le 1\\
&\le C_{13}\, e^{Ir\kappa(e(e-1))^{-1}t} \int_{\Omega_\varepsilon^p} \left(1+\left(I^{\frac{1}{2}}|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right)dx\\
&\le C_{13}\, e^{Ir\kappa(e(e-1))^{-1}t} \left(1+\left(I^{\frac{1}{2}}|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right)|\Omega| \quad \text{since } |\Omega_\varepsilon^p|<|\Omega|\\
&\le \sup_{\varepsilon>0} C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t}|\Omega| \left(1+\left(I^{\frac{1}{2}}|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right)\\
&\le \operatorname*{ess\,sup}_{t\in(0,T)} \sup_{\varepsilon>0} C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t}|\Omega| \left(1+\left(I^{\frac{1}{2}}|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right)\\
&\le \operatorname*{ess\,sup}_{t\in(0,T)} \sup_i \sup_{\varepsilon>0} C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t}|\Omega| \left(1+\left(I^{\frac{1}{2}}|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\sum_{i=1}^I ||u_{\varepsilon_i}(t)||^r_{L^r(\Omega_\varepsilon^p)} &\le I \operatorname*{ess\,sup}_{t\in(0,T)} \sup_i \sup_{\varepsilon>0} C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t}|\Omega| \left(1+\left(I^{\frac{1}{2}}|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{r(1+\alpha)}\right)\\
&= C_{14}^r,
\end{aligned} \tag{4.1.64}$$

where C_{14} is independent of i, ε and t but depends on r. For every $r \in \mathbb{N}$ $(2 \leq r < \infty)$, $C_{14}^r < \infty$ for all i, ε and t. Therefore

$$
\begin{aligned}
& |||u_\varepsilon(t)|||_{L^r(\Omega_\varepsilon^p)^I} \leq C_{14} < \infty \quad \forall\ \varepsilon,\ r \text{ and for a.e. } t \\
\Longrightarrow \quad & \sup_{\varepsilon>0} |||u_\varepsilon(t)|||_{L^r(\Omega_\varepsilon^p)^I} \leq C_{14} < \infty \quad \forall\ r \text{ and for a.e. } t\ .
\end{aligned}
$$

This establishes the inequality (4.1.60). Again from theorem 4.1.1.2.3, for $0 \leq t \leq T$, we have

$$
F_r(u_\varepsilon(t)) \leq e^{Ir\kappa(e(e-1))^{-1}t}\, F_r(\lambda u_0) \qquad \text{for all } r \text{ and for a.e. } t
$$

Proceeding as above, we obtain

$$
\begin{aligned}
||u_{\varepsilon_i}(t)||^r_{L^r(\Omega_\varepsilon^p)} &\leq C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t} \int_{\Omega_\varepsilon^p} \left(1+|u_0|_I^{r(1+\alpha)}\right) dx \\
&\leq C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t} \int_{\Omega_\varepsilon^p} \left(1+|u_0|_I^{(1+\alpha)}\right)^r dx \\
&= C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t} \left|\left|1+|u_0|_I^{(1+\alpha)}\right|\right|^r_{L^r(\Omega_\varepsilon^p)} \\
\Longrightarrow ||u_{\varepsilon_i}(t)||_{L^r(\Omega_\varepsilon^p)} &\leq \left(C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t}\right)^{\frac{1}{r}} \left|\left|1+|u_0|_I^{(1+\alpha)}\right|\right|_{L^r(\Omega_\varepsilon^p)} \\
&\leq \sup_{r\in\mathbb{N}} \left(C_{13}\ e^{Ir\kappa(e(e-1))^{-1}t}\right)^{\frac{1}{r}} \left|\left|1+|u_0|_I^{(1+\alpha)}\right|\right|_{L^r(\Omega_\varepsilon^p)} \forall\ i,\ r, \text{ and for a.e. } t.
\end{aligned}
\tag{4.1.65}
$$

Taking limit sup as $r \to \infty$ on both sides, we obtain

$$
\begin{aligned}
||u_{\varepsilon_i}(t)||_{L^\infty(\Omega_\varepsilon^p)} \leq \left|\left|1+|u_0|_I^{(1+\alpha)}\right|\right|_{L^\infty(\Omega_\varepsilon^p)} &\leq \operatorname*{ess\,sup}_{x\in\Omega_\varepsilon^p} \left(1+|u_0|_I^{(1+\alpha)}\right) \\
&\leq \left(1+\left(I^{\frac{1}{2}}\,|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{(1+\alpha)}\right) \\
&\leq \sup_{\varepsilon>0} \left(1+\left(I^{\frac{1}{2}}\,|||u_0|||_{L^\infty(\Omega_\varepsilon^p)^I}\right)^{(1+\alpha)}\right) \\
&= C_{15} < \infty \qquad \forall\ \varepsilon,\ i \text{ and for a.e. } t,
\end{aligned}
$$

i.e.,

$$
\max_{1\leq i\leq I} ||u_{\varepsilon_i}(t)||_{L^\infty(\Omega_\varepsilon^p)} \leq C_{15} < \infty \qquad \forall\ \varepsilon \text{ and for a.e. } t,
$$

i.e.,

$$
\sup_{\varepsilon>0} |||u_\varepsilon(t)|||_{L^\infty(\Omega_\varepsilon^p)^I} \leq C_{15} < \infty \qquad \text{for a.e. } t.
$$

♦

Corollary 4.1.1.2.9. *Let $p > n+2$, $r \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. Then there exists a positive constant C (depending only on $r \in \mathbb{N}$, T, $|\Omega|$ and I but independent of ε, λ and u_ε) such that any arbitrary solution $u_\varepsilon \in \mathcal{F}_\varepsilon^u$ of the problem $(P^{1^+}_{\varepsilon_{\lambda_M}})$ satisfies*

$$
|||u_\varepsilon|||_{\mathcal{F}_\varepsilon^u} \leq C.
$$

Proof. Choosing $r \in \mathbb{N}$ sufficiently large in corollary 4.1.1.2.8 and application of Hölder's inequality leads to the fact that the r.h.s., $\lambda S\bar{R}(u_\varepsilon)+\lambda\kappa u_\varepsilon$, of $(P^{1^+}_{\varepsilon_\lambda})$ is in $L^p((0,T);L^p(\Omega^p_\varepsilon))^I$. Since $L^p(\Omega^p_\varepsilon) \hookrightarrow H^{1,q}(\Omega^p_\varepsilon)^*$, $\lambda S\bar{R}(u_\varepsilon)+\lambda\kappa u_\varepsilon \in L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^I$.

The reformulation of (4.1.24)-(4.1.27) is given by

$$\frac{\partial u_\varepsilon(t)}{\partial t} + Au_\varepsilon(t) = f(t), \tag{4.1.66}$$

$$u_\varepsilon(0,x) = \lambda u_0(x), \tag{4.1.67}$$

where $f(t) = \lambda S\bar{R}(u_\varepsilon(t)) + \lambda\kappa u_\varepsilon(t)$ and $\kappa > 0$. The operator A is defined as in remark 4.1.1.1.1 and has the *maximal parabolic regularity* on $[H^{1,q}(\Omega^p_\varepsilon)^*]^I$. f is in $L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^I$. Moreover, by assumption (4.1.3), $u_0 \in \mathcal{X}^u_{p_\varepsilon}$. Therefore by theorem 3.3.1, there exists a $\tilde{C} > 0$ such that[16]

$$\begin{aligned}
|||u_\varepsilon|||_{\mathcal{F}^u_\varepsilon} &\le \tilde{C}\left(||\lambda u_0||_{\mathcal{X}^u_{p_\varepsilon}} + \left|\left|\lambda S\bar{R}(u_\varepsilon)+\lambda\kappa u_\varepsilon\right|\right|_{L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^I}\right)\\
&\le \tilde{C}\sup_{\varepsilon,\,\lambda>0}\left(||u_0||_{\mathcal{X}^u_{p_\varepsilon}} + \left|\left|S\bar{R}(u_\varepsilon)\right|\right|_{L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^I} + \kappa\,||u_\varepsilon||_{L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^I}\right)\\
&=: C < \infty,
\end{aligned}$$

where C is independent of ε, λ and u_ε. ♦

4.1.1.3 Compactness and Continuity of Z_1

Lemma 4.1.1.3.1. *The fixed point operator Z_1 is continuous and compact.*

Proof. Here we will only show the continuity of the operator Z_1 as the compactness follows with similar arguments. Let $(v_{\varepsilon_n})_{n\ge1}$ be a sequence in $\mathcal{F}^u_\varepsilon$ converging to a limit $v_\varepsilon \in \mathcal{F}^u_\varepsilon$. From theorem 3.4.3.4, $(v_{\varepsilon_n})_{n\ge1}$ is convergent to v_ε in $[L^\infty((0,T)\times\Omega^p_\varepsilon)]^I$. This implies that $(SR(v_{\varepsilon_n})+\kappa v_{\varepsilon_n})_{n\ge1}$ is convergent to $SR(v_\varepsilon)+\kappa v_\varepsilon$ in $[L^p((0,T)\times\Omega^p_\varepsilon)]^I$. Due to the continuous embedding $L^p(\Omega^p_\varepsilon)\hookrightarrow H^{1,q}(\Omega^p_\varepsilon)^*$, $(SR(v_{\varepsilon_n})+\kappa v_{\varepsilon_n})_{n\ge1}$ is convergent to $SR(v_\varepsilon)+\kappa v_\varepsilon$ in $[L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)]^I$. From theorem 3.3.1, we conclude that the map Z_1 is continuous. ♦

4.1.1.4 Existence and Uniqueness of the Solution

Proof of theorem 4.1.1.1. Applying Schaefer's fixed point theorem, thanks to corollary 4.1.1.2.9 and lemma 4.1.1.3.1, we get the existence of at least one fixed point, i.e., the existence of at least one solution of the problem $(P^{1^+}_{\varepsilon_M})$. This solution is also a solution of $(P^{1^+}_\varepsilon)$. Due to lemma 4.1.1.2, the solution of $(P^{1^+}_\varepsilon)$ solves (P^1_ε). Now we prove the uniqueness of the solution of (P^1_ε). Let u_{ε_1} and $u_{\varepsilon_2} \in \mathcal{F}^u_\varepsilon$ be two solutions of the problem (P^1_ε), where $u_{\varepsilon_1} \ne u_{\varepsilon_2}$. Set $\bar{u}_\varepsilon = u_{\varepsilon_1} - u_{\varepsilon_2}$. Then we have

$$\frac{\partial u_{\varepsilon_k}}{\partial t} - \nabla\cdot D\nabla u_{\varepsilon_k} = SR(u_{\varepsilon_k}) \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.1.68}$$

$$u_{\varepsilon_k}(0,x) = u_0(x) \quad \text{in } \Omega^p_\varepsilon, \tag{4.1.69}$$

$$-D\nabla u_{\varepsilon_k}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{4.1.70}$$

$$-D\nabla u_{\varepsilon_k}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.1.71}$$

[16]Note that $0 \le \lambda \le 1$.

for $k = 1,2$. Taking the difference and using $\bar{u}_{\varepsilon_i}$ as the test function in the $i-th$ PDE, we obtain

$$\frac{1}{2}\int_0^t \frac{d}{d\theta}||\bar{u}_{\varepsilon_i}(\theta)||^2_{L^2(\Omega^p_\varepsilon)}\,d\theta + D\int_0^t ||\nabla\bar{u}_{\varepsilon_i}(\theta)||^2_{L^2(\Omega^p_\varepsilon)}\,d\theta$$
$$\leq \frac{1}{2}\int_0^t \left[||SR(u_{\varepsilon_1}(\theta))_i - SR(u_{\varepsilon_2}(\theta))_i||^2_{L^2(\Omega^p_\varepsilon)} + ||\bar{u}_{\varepsilon_i}(\theta)||^2_{L^2(\Omega^p_\varepsilon)}\right] d\theta.$$

Expanding the term $R_j(u_{\varepsilon_1}) - R_j(u_{\varepsilon_2})$, each term in $R_j(u_{\varepsilon_1}) - R_j(u_{\varepsilon_2})$ contains a factor of the type $u_{\varepsilon_{1_l}} - u_{\varepsilon_{2_l}}$, whereas all the other factors are bounded in $L^\infty((0,T)\times\Omega^p_\varepsilon)$, therefore we obtain

$$||\bar{u}_{\varepsilon_i}(t)||^2_{L^2(\Omega^p_\varepsilon)} \leq C\int_0^t \sum_{i=1}^{I} ||\bar{u}_{\varepsilon_i}(\theta)||^2_{L^2(\Omega^p_\varepsilon)}\,d\theta,$$

i.e.,

$$|||\bar{u}_\varepsilon(t)|||^2_{L^2(\Omega^p_\varepsilon)^I} \leq C\int_0^t |||\bar{u}_\varepsilon(\theta)|||^2_{L^2(\Omega^p_\varepsilon)^I}\,d\theta.$$

Gronwall's inequality gives

$$|||\bar{u}_\varepsilon(t)|||^2_{L^2(\Omega^p_\varepsilon)^I} = 0 \quad \text{for a.e. } t,$$
$$\implies \quad u_{\varepsilon_1} = u_{\varepsilon_2}.$$

Hence the solution exists uniquely. ♦

Conclusion: We have shown the existence of a unique positive global weak solution of the problem (P^1_ε) in the first section 4.1.1. This has also provided us some very useful *a-priori* estimates (see (4.1.60) and (4.1.61)) with the help of a *Lyapunov functional*. These estimates will be further used during the homogenization of model M1, which is the main concern of the next section.

4.1.2 Homogenization of the Problem (P^1_ε)

4.1.2.1 A-priori Estimates

The aim of this section is to obtain ε - independent *a-priori* estimates for the solution u_ε of the micromodel in the domain $(0,T)\times\Omega^p_\varepsilon$ and then to extend these estimates to all of $(0,T)\times\Omega$. The major theorem of this section reads as:

Theorem 4.1.2.1.1. *There exists an extension of the solution u_ε to all of $(0,T)\times\Omega$ such that*

$$|||u_\varepsilon|||_{L^r((0,T);L^r(\Omega))^I} + |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega))^I} + |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega))^I} \leq C_{16}, \tag{4.1.72}$$

where C_{16} is independent of ε but depends only on r.

We start with the following lemma:

Lemma 4.1.2.1.2. *Let $p > n+2$ be fixed and $r \in \mathbb{N}$. Assume further that $u_\varepsilon \in \mathcal{F}^u_\varepsilon$ is the solution of the problem (P^1_ε), then we have the following estimate*

$$|||u_\varepsilon|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^I} + |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^I} + |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^I} \leq C_{17}, \tag{4.1.73}$$

where C_{17} is independent of ε but depends only on r.

Proof. The proof of this lemma consists of several steps.
(a) For $\lambda = 1$, the *a-priori* estimates obtained in (4.1.60) and (4.1.61) correspond to the *a-priori* estimates of the solution of the problem (P^1_ε)[17]. Therefore from (4.1.60), we have

$$\begin{aligned} |||u_\varepsilon(t)|||_{L^r(\Omega^p_\varepsilon)^I} &\leq C_{14} \quad \text{for all } r \text{ and for a.e. } t \\ \Longrightarrow \sum_{i=1}^{I} \int_0^T ||u_{\varepsilon_i}(t)||^r_{L^r(\Omega^p_\varepsilon)}\,dt &\leq \int_0^T C^r_{14}\,dt \\ \Longrightarrow |||u_\varepsilon|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^I} &\leq C_{18} \quad \text{for all } r, \end{aligned} \tag{4.1.74}$$

where C_{18} $(:= (C^r_{14}T)^{\frac{1}{r}})$ is independent of ε. Next,

$$\begin{aligned} |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^I} &= \max_{1\leq i\leq I} ||u_{\varepsilon_i}||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))} \\ &= \max_{1\leq i\leq I} \operatorname*{ess\,sup}_{t\in(0,T)} \operatorname*{ess\,sup}_{x\in\Omega^p_\varepsilon} |u_{\varepsilon_i}(t,x)| \\ &= \operatorname*{ess\,sup}_{t\in(0,T)} \max_{1\leq i\leq I} \operatorname*{ess\,sup}_{x\in\Omega^p_\varepsilon} |u_{\varepsilon_i}(t,x)| \\ &= \operatorname*{ess\,sup}_{t\in(0,T)} |||u_\varepsilon(t)|||_{L^\infty(\Omega^p_\varepsilon)^I} \\ &\leq \operatorname*{ess\,sup}_{t\in(0,T)} C_{15} \quad \text{by (4.1.61)} \\ &= C_{15}, \end{aligned} \tag{4.1.75}$$

where C_{15} is independent of ε.

(b) Testing the *i-th* PDE of (2.5.16) with $u_{\varepsilon_i}(t)$, we obtain[18]

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, u_{\varepsilon_i}(t)\right\rangle_{H^{1,q}(\Omega^p_\varepsilon)^*\times H^{1,q}(\Omega^p_\varepsilon)} dx\,dt \\ &\qquad - \int_0^T \langle \nabla\cdot D\nabla u_{\varepsilon_i}(t), u_{\varepsilon_i}(t)\rangle_{H^{1,q}(\Omega^p_\varepsilon)^*\times H^{1,q}(\Omega^p_\varepsilon)}\,dx\,dt \\ &\qquad\qquad = \int_0^T \langle SR(u_\varepsilon(t))_i, u_{\varepsilon_i}(t)\rangle_{H^{1,q}(\Omega^p_\varepsilon)^*\times H^{1,q}(\Omega^p_\varepsilon)}\,dt, \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{1}{2}\int_0^T \frac{d}{dt} ||u_{\varepsilon_i}(t)||^2_{L^2(\Omega^p_\varepsilon)}\,dt + \int_0^T D\,||\nabla u_{\varepsilon_i}(t)||^2_{L^2(\Omega^p_\varepsilon)}\,dt \\ &\qquad = \int_0^T \langle SR(u_\varepsilon(t))_i, u_{\varepsilon_i}(t)\rangle_{L^p(\Omega^p_\varepsilon)\times L^q(\Omega^p_\varepsilon)}\,dt \\ &\qquad \leq \frac{1}{p}\int_0^T ||SR(u_\varepsilon(t))_i||^p_{L^p(\Omega^p_\varepsilon)}\,dt + \frac{1}{q}\int_0^T ||u_{\varepsilon_i}(t)||^q_{L^q(\Omega^p_\varepsilon)}\,dt, \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{1}{2}||u_{\varepsilon_i}(T)||^2_{L^2(\Omega^p_\varepsilon)} + \int_0^T D\,||\nabla u_{\varepsilon_i}(t)||^2_{L^2(\Omega^p_\varepsilon)}\,dt \\ &\leq \frac{1}{2}||u_{0_i}||^2_{L^2(\Omega^p_\varepsilon)} + \frac{1}{p}\int_0^T ||SR(u_\varepsilon(t))_i||^p_{L^p(\Omega^p_\varepsilon)}\,dt + \frac{1}{q}\int_0^T ||u_{\varepsilon_i}(t)||^q_{L^q(\Omega^p_\varepsilon)}\,dt. \end{aligned} \tag{4.1.76}$$

[17] See the remark after the theorem 9.2.2.4 in [Eva98].

[18] From (4.1.60), we have $||u_{\varepsilon_i}(t)||_{L^r(\Omega^p_\varepsilon)} \leq C$ for all i and for a.e. t, where C is independent of ε. This gives $||SR(u_\varepsilon)_i||_{L^p(\Omega^p_\varepsilon)} \leq C$. Since $L^p(\Omega^p_\varepsilon) \hookrightarrow H^{1,q}(\Omega^p_\varepsilon)^*$, from the definition (3.1.3) we get $\langle SR(u_\varepsilon)_i, \phi_i\rangle_{H^{1,q}(\Omega^p_\varepsilon)^*\times H^{1,q}(\Omega^p_\varepsilon)} = \langle SR(u_\varepsilon)_i, \phi_i\rangle_{L^p(\Omega^p_\varepsilon)\times L^q(\Omega^p_\varepsilon)}$ for $\phi_i \in H^{1,q}(\Omega^p_\varepsilon)$.

Choosing r in (4.1.60) sufficiently large such that $\sup\limits_{\varepsilon>0} ||u_{\varepsilon_i}(t)||^q_{L^q(\Omega^p_\varepsilon)} < \infty$ and $\sup\limits_{\varepsilon>0} ||SR(u_\varepsilon(t))_i||^p_{L^p(\Omega^p_\varepsilon)} < \infty$. Also from theorem 3.4.3.3, it follows that $\sup\limits_{\varepsilon>0} ||u_{0_i}||^2_{L^2(\Omega^p_\varepsilon)} < \infty$. [19] Therefore the r.h.s. of (4.1.76) is bounded by a constant independent of ε, i and t. Let us call this constant by $\bar{C}$. This gives

$$
\begin{aligned}
& \int_0^T D\,||\nabla u_{\varepsilon_i}(t)||^2_{L^2(\Omega^p_\varepsilon)}\,dt && \leq && \bar{C} \quad \text{for all } \varepsilon,\ i \text{ and for a.e. t} \\
\Longrightarrow\quad & \sum_{i=1}^{I}\int_0^T ||\nabla u_{\varepsilon_i}(t)||^2_{L^2(\Omega^p_\varepsilon)}\,dt && \leq && \sum_{i=1}^{I}\frac{\bar{C}}{D} \\
\Longrightarrow\quad & |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^I} && \leq && C_{19}, \\
\Longrightarrow\quad & \sup_{\varepsilon>0} |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^I} && \leq && C_{19},
\end{aligned}
\tag{4.1.77}
$$

where C_{19} $(:= \left(\frac{\bar{C}}{D} I\right)^{\frac{1}{2}})$ is independent of ε. Note that $D > 0$ is a constant. Adding (4.1.74), (4.1.75) and (4.1.77) will yield

$$
\begin{aligned}
& |||u_\varepsilon|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^I} + |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^I} + |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^I} \\
& \leq\ C_{18} + C_{15} + C_{19} \quad \text{for all } r \\
& =\ C_{17} \quad \text{for all } r,
\end{aligned}
$$

where C_{17} $(:= C_{18} + C_{15} + C_{19})$ is independent of ε but depends only on r. ♦

Proof of theorem 4.1.2.1.1: The estimate (4.1.73) from the lemma 4.1.2.1.2 and the theorem 3.4.2.3 accomplish the proof. ♦

4.1.2.2 Convergence of the Micro Solution

In this subsection we show the *weak*, *strong* and *two-scale* convergences of the solution of the microproblem (P^1_ε).

Theorem 4.1.2.2.1. *There exists a constant C_{20} such that the solution, u_ε, of the problem (P^1_ε) satisfies the following estimate:*

$$
|||u_\varepsilon|||_{L^\infty((0,T);L^2(\Omega))^I} + |||u_\varepsilon|||_{L^2((0,T);H^{1,2}(\Omega))^I} + \left|\left|\left|\chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^I} \leq C_{20},
\tag{4.1.78}
$$

where C_{20} is independent of ε but depends only on r.

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$$
\begin{aligned}
||u_{0_i}||^2_{L^2(\Omega^p_\varepsilon)} \leq \left|\Omega^p_\varepsilon\right| ||u_{0_i}||^2_{L^\infty(\Omega^p_\varepsilon)} & \leq C\,|\Omega|\,||u_{0_i}||^2_{(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} \quad \text{by theorem 3.4.3.3} \\
& \leq C\,|\Omega| \sup_{\varepsilon>0} ||u_{0_i}||^2_{(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} \\
& \leq C\,|\Omega| < \infty \ \forall\, \varepsilon \\
\Longrightarrow \sup_{\varepsilon>0} ||u_{0_i}||^2_{L^2(\Omega^p_\varepsilon)} \leq C\,|\Omega| < \infty. &
\end{aligned}
$$

Proof. The proof consists of several steps.

(a)
$$
\begin{aligned}
|||u_\varepsilon|||^2_{L^\infty((0,T);L^2(\Omega))^I} &= \max_{1\le i\le I} \operatorname*{ess\,sup}_{t\in(0,T)} ||u_{\varepsilon_i}(t)||^2_{L^2(\Omega)} \\
&= \max_{1\le i\le I} \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |u_{\varepsilon_i}(t,x)|^2\, dx \\
&\le \max_{1\le i\le I} \operatorname*{ess\,sup}_{t\in(0,T)} \operatorname*{ess\,sup}_{x\in\Omega} |u_{\varepsilon_i}(t,x)|^2\, |\Omega| \\
&= |\Omega|\, |||u_\varepsilon|||^2_{L^\infty((0,T);L^\infty(\Omega))^I} \\
&\le |\Omega|\, C_{16}^2 \quad \text{by (4.1.72)},
\end{aligned}
$$

i.e.,

$$
|||u_\varepsilon|||_{L^\infty((0,T);L^2(\Omega))^I} \le C_{21}, \tag{4.1.79}
$$

where C_{21} $(:= (C_{16}^2\,|\Omega|)^{\frac{1}{2}})$ is independent of ε.

(b)
$$
\begin{aligned}
|||u_\varepsilon|||^2_{L^2((0,T);H^{1,2}(\Omega))^I} &= \sum_{i=1}^{I} ||u_{\varepsilon_i}||^2_{L^2((0,T);H^{1,2}(\Omega))} \\
&= \sum_{i=1}^{I} \Big(||\nabla u_{\varepsilon_i}||^2_{L^2((0,T);L^2(\Omega))} + ||u_{\varepsilon_i}||^2_{L^2((0,T);L^2(\Omega))} \Big) \\
&\le \sup_{\varepsilon>0} \sum_{i=1}^{I} \Big(||\nabla u_{\varepsilon_i}||^2_{L^2((0,T);L^2(\Omega))} + (T\,|\Omega|)^{1-\frac{2}{r}}\, ||u_{\varepsilon_i}||^2_{L^r((0,T);L^r(\Omega))} \Big) \\
&=: C_{22} < \infty, \quad \text{by (4.1.72)},
\end{aligned}
$$

i.e.,

$$
|||u_\varepsilon|||_{L^2((0,T);H^{1,2}(\Omega))^I} \le C_{22}, \tag{4.1.80}
$$

where C_{22} is independent of ε but depends only on r.
(c) Let $\phi \in H_0^{1,2}(0,T)$ and $\psi \in H^{1,2}(\Omega)$. Then the weak formulation of the i-*th* PDE of the problem (2.5.16)-(2.5.19) is given by

$$
\begin{aligned}
&\int_0^T \langle \chi^\varepsilon \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\, dt \\
&\qquad + \int_0^T \int_\Omega \phi(t)\chi^\varepsilon(x)\nabla u_{\varepsilon_i}(t,x)\nabla\psi(x)\, dx\, dt \\
&\qquad\quad = \int_0^T \langle \chi^\varepsilon SR(u_\varepsilon(t))_i, \phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\, dt\, dt,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\left| \int_0^T \langle \chi^\varepsilon \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\, dt \right| \\
&\qquad \le \int_0^T \int_\Omega |\chi^\varepsilon(x)|\,|\nabla u_{\varepsilon_i}(t,x)|\,|\nabla\psi(x)|\,|\phi(t)|\, dx\, dt \\
&\qquad\quad + \frac{1}{2}\int_0^T \Big[||\chi^\varepsilon SR(u_\varepsilon(t))_i||^2_{L^2(\Omega)} + ||\phi(t)\psi||^2_{L^2(\Omega)} \Big]\, dt.
\end{aligned}
$$

Note that $|\chi^\varepsilon(x)| \le 1$. From (4.1.72) the terms $\sup\limits_{\varepsilon>0} ||\nabla u_{\varepsilon_i}||^2_{L^2((0,T);L^2(\Omega))}$ and $\sup\limits_{\varepsilon>0} ||SR(u_\varepsilon)_i||^2_{L^2((0,T);L^2(\Omega))}$ are finite. This gives

$$\begin{aligned}
&\left| \int_0^T \langle \chi^\varepsilon \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\, dt \right| \\
&\qquad \le\; C + \frac{1}{2}||\phi(t)||^2_{L^2(0,T)} \left[||\nabla\psi||^2_{L^2(\Omega)} + ||\psi||^2_{L^2(\Omega)} \right] \\
&\qquad =\; C + ||\phi||^2_{L^2(0,T)}\, ||\psi||^2_{H^{1,2}(\Omega)}\, .
\end{aligned}$$

$\phi \in H_0^{1,2}(0,T)$ implies $||\phi||_{L^2(0,T)} \le \bar{C}\,||\phi||_{H_0^{1,2}(0,T)}$, i.e., $\left|\left|\frac{\phi}{\bar{C}}\right|\right|_{L^2(0,T)} \le ||\phi||_{H_0^{1,2}(0,T)}$, where $\bar{C} > 0$ is the embedding constant. Taking the supremum on both sides,

$$\begin{aligned}
&\bar{C} \sup_{\substack{\frac{\phi}{\bar{C}}\in L^2(0,T)\\ \psi\in H^{1,2}(\Omega)\\ ||\frac{\phi}{\bar{C}}||^2_{L^2(0,T)}\le 1\\ ||\psi||^2_{H^{1,2}(\Omega)}\le 1}} \left| \int_0^T \langle \chi^\varepsilon \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \frac{\phi(t)}{\bar{C}}\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\, dt \right| \\
&\qquad \le C + \bar{C}^2 \sup_{\substack{\frac{\phi}{\bar{C}}\in L^2(0,T)\\ \psi\in H^{1,2}(\Omega)\\ ||\frac{\phi}{\bar{C}}||^2_{L^2(0,T)}\le 1\\ ||\psi||^2_{H^{1,2}(\Omega)}\le 1}} ||\psi||^2_{H^{1,2}(\Omega)} \left|\left|\frac{\phi}{\bar{C}}\right|\right|^2_{L^2(0,T)} .
\end{aligned}$$

This implies

$$\left|\left| \chi^\varepsilon \frac{\partial u_{\varepsilon_i}}{\partial t} \right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)} \le C_{23}$$

$$\begin{aligned}
&\Longrightarrow \sum_{i=1}^{I} \left|\left| \chi^\varepsilon \frac{\partial u_{\varepsilon_i}}{\partial t} \right|\right|^2_{L^2((0,T);H^{1,2}(\Omega)^*)} \le\; I\, C_{23}^2 \\
&\Longrightarrow \left|\left|\left| \chi^\varepsilon \frac{\partial u_{\varepsilon}}{\partial t} \right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^I} \le\; C_{24},
\end{aligned} \tag{4.1.81}$$

where C_{24} $(:= (I\, C_{23}^2)^{\frac{1}{2}})$ is independent of ε but depends only on r. Adding (4.1.79), (4.1.80) and (4.1.81), we obtain

$$\begin{aligned}
&||u_\varepsilon||_{L^\infty((0,T);L^2(\Omega))^I} + ||u_\varepsilon||_{L^2((0,T);H^{1,2}(\Omega))^I} + \left|\left| \chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^I} \\
&\le\; C_{21} + C_{22} + C_{24} \\
&=\; C_{20},
\end{aligned}$$

where C_{20} $(:= C_{21} + C_{22} + C_{24})$ is independent of ε but depends only on r. ♦

The next statement is very crucial. It gives the strong convergence of the subsequence of the sequence $(u_{\varepsilon_i})_{\varepsilon>0}$. This is the main result of Meirmanov & Zimin in [MZ11].

Lemma 4.1.2.2.2. *Let $(c_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^{1,2}(\Omega))$ and weakly convergent in $L^2((0,T);L^2(\Omega))\cap L^2((0,T);H^{1,2}(\Omega))$ to a function c. Suppose further that the sequence $(\frac{\partial}{\partial t}\chi^\varepsilon c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T);H^{1,2}(\Omega)^*)$. Then the sequence $(c_\varepsilon)_{\varepsilon>0}$ is strongly convergent to the function c in $L^2((0,T);L^2(\Omega))$.*

Proof. See theorem 2.1 in Meirmanov & Zimin [MZ11]. ♦

Theorem 4.1.2.2.3. *Let $(u_\varepsilon)_{\varepsilon>0}$ satisfies the estimates (4.1.72) and (4.1.78). Then there exists a function $u\in L^2((0,T);H^{1,2}(\Omega))^I$ and a function $u^1\in L^2((0,T)\times\Omega;H^{1,2}_{per}(Y)/\mathbb{R})^I$ such that up to a subsequence, still denoted by same subscript, the following convergence results hold:*

$$(i)\ (u_\varepsilon)_{\varepsilon>0} \text{ is weakly convergent to } u \text{ in } L^2((0,T);H^{1,2}(\Omega))^I. \tag{4.1.82}$$

$$(ii)\ (u_\varepsilon)_{\varepsilon>0} \text{ is strongly convergent to } u \text{ in } L^2((0,T);L^2(\Omega))^I. \tag{4.1.83}$$

$$(iii)\ (u_\varepsilon)_{\varepsilon>0} \text{ and } (\nabla_x u_\varepsilon)_{\varepsilon>0} \text{ are two-scale convergent to } u \text{ and } \nabla_x u+\nabla_y u^1 \text{ in the sense of (3.5.3) respectively.} \tag{4.1.84}$$

Proof. (i) From the estimate (4.1.78), we note that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T);H^{1,2}(\Omega))^I$. This implies that, up to a subsequnce, still indexed by the same subscript, $(u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to a function u in $L^2((0,T);H^{1,2}(\Omega))^I$.
(ii) From (4.1.78), it follows that, up to a subsequence, still denoted by the same subscript, $(u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to u in $L^2((0,T);L^2(\Omega))^I\cap L^2((0,T);H^{1,2}(\Omega))^I$ and is bounded in $L^\infty((0,T);L^2(\Omega))^I\cap L^2((0,T);H^{1,2}(\Omega))^I$. Also from (4.1.78) note that $(\frac{\partial}{\partial t}\chi^\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T);H^{1,2}(\Omega)^*)^I$. Therefore the subsequence $(u_\varepsilon)_{\varepsilon>0}$, still denoted by the same subscript, is strongly convergent to u in $L^2((0,T);L^2(\Omega))^I$.
(iii) The proof follows from the estimate (4.1.78) and theorem 3.5.13. ♦

Theorem 4.1.2.2.4. *The limit function u belongs to $L^\infty((0,T)\times\Omega\times Y)^I$.*[20]

Proof. Since $(u_\varepsilon)_{\varepsilon>0}$ is strongly convergent to u in $L^2((0,T);L^2(\Omega))^I$, there exists a subsequence $(u_{\varepsilon'})_{\varepsilon'>0}$ which is pointwise convergent[21] to u almost everywhere in $(0,T)\times\Omega$, i.e.,

$$\lim_{\varepsilon'\to 0} u_{\varepsilon'}(t,x)=u(t,x) \qquad \text{a.e.} \qquad (t,x)\in(0,T)\times\Omega.$$

By theorem 4.1.2.1.1, we have $||u_{\varepsilon_i}||_{L^\infty((0,T);L^\infty(\Omega))}\le C_{16}$ for all i, therefore

$$\begin{aligned}
|u_i(t,x)|^2 \le |u(t,x)|_I^2 = \sum_{i=1}^I |u_i(t,x)|^2 = \lim_{\varepsilon'\to 0}\sum_{i=1}^I \left|u_{\varepsilon_i'}(t,x)\right|^2 &\\
\le \sum_{i=1}^I \limsup_{\varepsilon'\to 0}\ \operatorname*{ess\,sup}_{t\in(0,T)}\ \operatorname*{ess\,sup}_{x\in\Omega}\ \left|u_{\varepsilon_i'}(t,x)\right|^2 &\\
\le \sum_{i=1}^I \limsup_{\varepsilon'\to 0}\ C_{16}^2 &\\
= C_{16}^2\, I \text{ for a.e. } t \text{ and } x &\\
\implies \operatorname*{ess\,sup}_{t\in(0,T)}\ \operatorname*{ess\,sup}_{x\in\Omega}\ |u_i(t,x)|^2 \le C_{16}^2\, I<\infty \quad \text{for all } i. &
\end{aligned}$$

[20]Note that the function u is independent of the variable y.
[21]Cf. corollary on page 53 in [Yos70].

This gives

$$\begin{aligned}|||u|||^2_{L^\infty((0,T)\times\Omega\times Y)^I} = \max_{1\le i\le I} ||u_i||^2_{L^\infty((0,T)\times\Omega\times Y)} &= \max_{1\le i\le I} \operatorname*{ess\,sup}_{(t,x,y)\in(0,T)\times\Omega\times Y} |u_i(t,x)|^2 \\ &\le \max_{1\le i\le I} \operatorname*{ess\,sup}_{y\in Y} \operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\Omega} |u_i(t,x)|^2 \\ &\le \max_{1\le i\le I} \operatorname*{ess\,sup}_{y\in Y} \operatorname*{ess\,sup}_{t\in(0,T)} \operatorname*{ess\,sup}_{x\in\Omega} |u_i(t,x)|^2 \\ &\le \operatorname*{ess\,sup}_{y\in Y} C_{16}^2\, I,\end{aligned}$$

i.e., $|||u|||_{L^\infty((0,T)\times\Omega\times Y)^I} \le C_{25}$, where C_{25} $(:= (C_{16}^2 I)^{\frac{1}{2}})$ is independent of ε but depends only on r. ♦

Corollary 4.1.2.2.5. *For all $2 \le p < \infty$, $(u_\varepsilon)_{\varepsilon>0}$ is strongly convergent to u in $L^p((0,T)\times\Omega)^I$.*

Proof. This follows from the straightforward application of Lyapunov's interpolation inequality (cf. lemma A.6) and L^∞ - estimates of u_ε and u. See lemma 3.2.20 in [Pet06] for details. ♦

Theorem 4.1.2.2.6. *The sequence $(SR(u_\varepsilon))_{\varepsilon>0}$ is strongly convergent to $SR(u)$ in $L^2((0,T)\times\Omega)^I$ as $\varepsilon \to 0$.*

Proof. Note that

$$|||SR(u_\varepsilon) - SR(u)|||^2_{L^2((0,T)\times\Omega)^I} = \sum_{i=1}^{I} ||SR(u_\varepsilon)_i - SR(u)_i||^2_{L^2((0,T)\times\Omega)} \tag{4.1.85}$$

From (2.4.7), we have

$$SR(u_\varepsilon)_i = \sum_{j=1}^{J} s_{ij} \left(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon m}^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon m}^{s_{mj}} \right) \tag{4.1.86}$$

and

$$SR(u)_i = \sum_{j=1}^{J} s_{ij} \left(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \right). \tag{4.1.87}$$

From (4.1.86) and (4.1.87),

$$\begin{aligned}||SR(u_\varepsilon)_i - SR(u)_i||_{L^2((0,T)\times\Omega)} = \Bigg\| &\sum_{j=1}^{J} s_{ij} \left(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon m}^{-s_{mj}} - k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} \right) \\ &- \sum_{j=1}^{J} s_{ij} \left(k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon m}^{s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \right) \Bigg\|_{L^2((0,T)\times\Omega)}\end{aligned}$$

$$\leq \sum_{j=1}^{J} s_{ij}k_j^f \left\| \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon_m}^{-s_{mj}} - \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} \right\|_{L^2((0,T)\times\Omega)}$$

$$+ \sum_{j=1}^{J} s_{ij}k_j^b \left\| \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon_m}^{s_{mj}} - \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \right\|_{L^2((0,T)\times\Omega)} . \tag{4.1.88}$$

It can be easily shown that the terms $\left\| \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon_m}^{-s_{mj}} - \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} \right\|_{L^2((0,T)\times\Omega)}$ and $\left\| \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon_m}^{s_{mj}} - \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \right\|_{L^2((0,T)\times\Omega)}$ are strongly convergent to 0 as $\varepsilon \to 0$.[22] Therefore $\|SR(u_\varepsilon)_i - SR(u)_i\|_{L^2((0,T)\times\Omega)} \to 0$ as $\varepsilon \to 0$. From (4.1.85), the theorem follows. ♦

Remark 4.1.2.2.7. The strong convergence of $(SR(u_\varepsilon))_{\varepsilon>0}$ implies that it is *two-scale* convergent to $SR(u)$ in the sense of (3.5.3).

4.1.2.3 Passage to the Limit as $\varepsilon \to 0$

Let us consider the functions $\phi_0 \in C_0^\infty((0,T)\times\Omega)^I$ and $\phi_1 \in C_0^\infty(((0,T)\times\Omega);C_{per}^\infty(Y))^I$ such that $\phi(t,x,\frac{x}{\varepsilon}) := \phi_0(t,x) + \varepsilon\phi_1(t,x,\frac{x}{\varepsilon}) \in C_0^\infty(((0,T)\times\Omega);C_{per}^\infty(Y))^I$. Using ϕ as test function in the weak formulation of (2.5.16)-(2.5.19) one obtains

$$\int_0^T \left\langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi(t) \right\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I\times[H^{1,2}(\Omega_\varepsilon^p)]^I} dt - \int_0^T \langle \nabla\cdot D\nabla u_\varepsilon(t), \phi(t)\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I\times[H^{1,2}(\Omega_\varepsilon^p)]^I} dt$$
$$= \int_0^T \langle SR(u_\varepsilon(t)), \phi(t)\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I\times[H^{1,2}(\Omega_\varepsilon^p)]^I} dt,$$

i.e.,

$$\sum_{i=1}^{I}\int_0^T \langle \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)} dt - \sum_{i=1}^{I}\int_0^T \langle \nabla\cdot D\nabla u_{\varepsilon_i}(t), \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)} dt$$
$$= \sum_{i=1}^{I}\int_0^T \langle SR(u_\varepsilon(t))_i, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)} dt,$$

i.e.,

$$\sum_{i=1}^{I}\int_0^T \langle \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)} dt + \sum_{i=1}^{I}\int_0^T \int_{\Omega_\varepsilon^p} D\nabla u_{\varepsilon_i}(t,x)\nabla\phi_i(t,x,\frac{x}{\varepsilon})\,dx\,dt$$
$$= \sum_{i=1}^{I}\int_0^T \langle SR(u_\varepsilon(t))_i, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)}\,dx\,dt. \tag{4.1.89}$$

Now we pass the *two-scale* limit in (4.1.89) term by term.

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T \langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)} dt$$
$$= \quad -\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T \int_{\Omega_\varepsilon^p} u_{\varepsilon_i}(t,x)\left(\frac{\partial\phi_{0_i}(t,x)}{\partial t} + \varepsilon\frac{\partial\phi_{1_i}(t,x,\frac{x}{\varepsilon})}{\partial t}\right) dx\,dt$$

[22]Since this is just a mere calculation, we intended not to include here.

$$
\begin{aligned}
&= -\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_\Omega \chi(\frac{x}{\varepsilon})u_{\varepsilon_i}(t,x)\frac{\partial\phi_{0_i}}{\partial t}\,dx\,dt - \underbrace{\lim_{\varepsilon\to 0}\varepsilon\sum_{i=1}^{I}\int_0^T\int_\Omega \chi(\frac{x}{\varepsilon})u_{\varepsilon_i}(t,x)\frac{\partial\phi_{1_i}}{\partial t}\,dx\,dt}_{=0}\\
&= -\sum_{i=1}^{I}\int_0^T\int_\Omega\int_Y \chi(y)u_i(t,x)\frac{\partial\phi_{0_i}(t,x)}{\partial t}\,dx\,dy\,dt\\
&= -\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p} u_i(t,x)\frac{\partial\phi_{0_i}(t,x)}{\partial t}\,dx\,dy\,dt,\ \text{since } \chi(y)=1 \text{ in } Y^p\\
&= |Y^p|\sum_{i=1}^{I}\int_0^T\left\langle\frac{\partial u_i(t)}{\partial t},\phi_{0_i}(t)\right\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)}dt\\
&= |Y^p|\int_0^T\left\langle\frac{\partial u(t)}{\partial t},\phi_0(t)\right\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I\times[H^{1,2}(\Omega_\varepsilon^p)]^I}dt. \qquad (4.1.90)
\end{aligned}
$$

Again,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_{\Omega_\varepsilon^p} D\nabla_x u_{\varepsilon_i}(t,x)\nabla_x\phi_i(t,x,\frac{x}{\varepsilon})\,dx\,dt\\
&= \lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_{\Omega_\varepsilon^p} D\nabla_x u_{\varepsilon_i}(t,x)\nabla_x\left(\phi_{0_i}(t,x)+\varepsilon\phi_{1_i}(t,x,\frac{x}{\varepsilon})\right)dx\,dt\\
&= \lim_{\varepsilon\to 0}\left[\sum_{i=1}^{I}\int_0^T\int_\Omega \chi(\frac{x}{\varepsilon})D\nabla_x u_{\varepsilon_i}(t,x)\left(\nabla_x\phi_{0_i}(t,x)+\nabla_y\phi_{1_i}(t,x,\frac{x}{\varepsilon})\right)dx\,dt\right.\\
&\qquad\left.+\varepsilon\sum_{i=1}^{I}\int_0^T\int_\Omega \chi(\frac{x}{\varepsilon})D\nabla_x u_{\varepsilon_i}(t,x)\nabla_x\phi_{1_i}(t,x,\frac{x}{\varepsilon})\,dx\,dt\right]\\
&= \lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_\Omega \chi(\frac{x}{\varepsilon})D\nabla_x u_{\varepsilon_i}(t,x)\left(\nabla_x\phi_{0_i}(t,x)+\nabla_y\phi_{1_i}(t,x,\frac{x}{\varepsilon})\right)dx\,dt\\
&\qquad+\underbrace{\lim_{\varepsilon\to 0}\varepsilon\sum_{i=1}^{I}\int_0^T\int_\Omega \chi(\frac{x}{\varepsilon})D\nabla_x u_{\varepsilon_i}(t,x)\nabla_x\phi_{1_i}(t,x,\frac{x}{\varepsilon})\,dx\,dt}_{=0}\\
&= \sum_{i=1}^{I}\int_0^T\int_\Omega\int_Y \chi(y)D\left(\nabla_x u_i(t,x)+\nabla_y u_{1_i}(t,x,y)\right)\left(\nabla_x\phi_{0_i}(t,x)+\nabla_y\phi_{1_i}(t,x,y)\right)dx\,dy\,dt\\
&= \sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p} D\left(\nabla_x u_i(t,x)+\nabla_y u_{1_i}(t,x,y)\right)\left(\nabla_x\phi_{0_i}(t,x)+\nabla_y\phi_{1_i}(t,x,y)\right)dx\,dy\,dt. \qquad (4.1.91)
\end{aligned}
$$

FInally,

$$
\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\langle SR(u_\varepsilon(t))_i,\phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)}\,dt
$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T \langle SR(u_\varepsilon(t))_i,\phi_i(t)\rangle_{L^2(\Omega_\varepsilon^p)\times L^2(\Omega_\varepsilon^p)}\,dt^{23}\\
&= \lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_{\Omega_\varepsilon^p} SR(u_\varepsilon(t,x))_i\phi_i(t,x)\,dx\,dt\\
&= \lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_{\Omega}\chi(\frac{x}{\varepsilon})SR(u_\varepsilon)_i\phi_{0_i}(t,x)\,dx\,dt+\underbrace{\lim_{\varepsilon\to 0}\varepsilon\sum_{i=1}^{I}\int_0^T\int_{\Omega}\chi(\frac{x}{\varepsilon})SR(u_\varepsilon)_i\phi_{1_i}(t,x,\frac{x}{\varepsilon})\,dx\,dt}_{=0}\\
&= \sum_{i=1}^{I}\int_0^T\int_\Omega\int_Y \chi(y)SR(u(t,x))_i\phi_{0_i}(t,x)\,dx\,dy\,dt\\
&= \sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p} SR(u(t,x))_i\phi_{0_i}(t,x)\,dx\,dy\,dt\\
&= |Y^p|\int_0^T\langle SR(u(t)),\phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt.
\end{aligned}
\tag{4.1.92}
$$

Combining (4.1.90), (4.1.91) and (4.1.92), we get

$$
\begin{aligned}
&|Y^p|\int_0^T\left\langle\frac{\partial u(t)}{\partial t},\phi_0(t)\right\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}dt\\
&\quad+\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p} D\left(\nabla_x u_i(t,x)+\nabla_y u_{1_i}(t,x,y)\right)\left(\nabla_x\phi_{0_i}(t,x)+\nabla_y\phi_{1_i}(t,x,y)\right)dx\,dy\,dt\\
&\qquad=|Y^p|\int_0^T\langle SR(u(t)),\phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt.
\end{aligned}
\tag{4.1.93}
$$

Now choosing $\phi_0(t,x)\equiv 0$, i.e., $\phi_{0_i}(t,x)\equiv 0$ for all $i=1,2,...,I$, then $\phi(t,x)=\phi_1(t,x,\frac{x}{\varepsilon})$ and the equation (4.1.93) reduces to

$$
\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p} D(\nabla_x u_i(t,x)+\nabla_y u_{1_i}(t,x,y))\nabla_y\phi_{1_i}(t,x,y)\,dx\,dy\,dt=0. \tag{4.1.94}
$$

Let us choose $u_{1_i}(t,x,y)=\sum_{j=1}^{n}\frac{\partial u_i(t,x)}{\partial x_j}a_j(t,x,y)+c_i(x)$, for all $i=1,2,...,I$, where $c(x)$ is any arbitrary function of x. The equation (4.1.94) is satisfied by each u_{1_i} if a_j, for $j=1,2,...,n$, is the solution of the *Cell-Problem*

$$
\begin{aligned}
-\nabla_y\cdot(D(\nabla_y a_j(t,x,y)+e_j))=0 &\quad \text{for } (t,x,y)\in(0,T)\times\Omega\times Y^p, &(4.1.95)\\
-D(\nabla_y a_j(t,x,y)+e_j)\cdot\vec{n}=0 &\quad \text{for } (t,x,y)\in(0,T)\times\Omega\times\Gamma, &(4.1.96)\\
y\mapsto a_j(y) \text{ is } Y-\text{periodic.} & &(4.1.97)
\end{aligned}
$$

On the other hand, if a_j is the solution of the cell-problem (4.1.95)-(4.1.97), the equation (4.1.94) is satisfied if $u_{1_i}(t,x,y)=\sum_{j=1}^{n}\frac{\partial u_i(t,x)}{\partial x_j}a_j(t,x,y)+c_i(x)$. Setting $\phi_1(t,x,\frac{x}{\varepsilon})\equiv 0$, i.e.,

[23] By (4.1.60), $\sup_{\varepsilon>0}||u_{\varepsilon_i}(t)||_{L^r(\Omega_\varepsilon^p)}\le C_{16}$ $\forall i$ and for a.e. t. This gives $\sup_{\varepsilon>0}||SR(u_\varepsilon)_i||_{L^2(\Omega_\varepsilon^p)}\le C$. Since $L^2(\Omega_\varepsilon^p)\hookrightarrow H^{1,2}(\Omega_\varepsilon^p)^*$, from (3.1.3) $\langle SR(u_\varepsilon)_i,\phi_i\rangle_{H^{1,2}(\Omega_\varepsilon^p)^*\times H^{1,2}(\Omega_\varepsilon^p)}=\langle SR(u_\varepsilon)_i,\phi_i\rangle_{L^2(\Omega_\varepsilon^p)\times L^2(\Omega_\varepsilon^p)}$, $\phi_i\in H^{1,2}(\Omega_\varepsilon^p)$.

$\phi_{1_i}(t,x,\frac{x}{\varepsilon}) \equiv 0$ for all i. Then the equation (4.1.93) reduces to

$$|Y^p| \int_0^T \left\langle \frac{\partial u(t)}{\partial t}, \phi_0(t) \right\rangle_{[H^{1,2}(\Omega)^*]^I \times [H^{1,2}(\Omega)]^I} dt$$
$$+ \sum_{i=1}^I \int_0^T \int_\Omega \int_{Y^p} D\left(\nabla_x u_i(t,x) + \nabla_y u_{1_i}(t,x,y)\right)\left(\nabla_x \phi_{0_i}(t,x) + \nabla_y \phi_{1_i}(t,x,y)\right) dx\, dy\, dt$$
$$= |Y^p| \int_0^T \langle SR(u(t)), \phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I \times [H^{1,2}(\Omega)]^I}\, dt,$$

i.e.,

$$\sum_{i=1}^I \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_{0_i}(t) \right\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} dt$$
$$+ \sum_{i=1}^I \int_0^T \int_\Omega \int_{Y^p} \frac{D}{|Y^p|}(\nabla_x u_i(t,x) + \nabla_y u_{1_i}(t,x,y)) \nabla_x \phi_{0_i}(t,x)\, dx\, dy\, dt$$
$$= \sum_{i=1}^I \int_0^T \langle SR(u(t))_i, \phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)}\, dt. \tag{4.1.98}$$

Substituting $u_{1_i}(t,x,y) = \vec{a}(t,x,y) \cdot \nabla_x u_i(t,x) + c(x)$, i.e., $\nabla_y u_{1_i} = \sum_{j=1}^n \nabla_y a_j \frac{\partial u_i}{\partial x_j}$ in (4.1.98), then we obtain

$$\sum_{i=1}^I \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_{0_i}(t) \right\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} dt$$
$$+ \sum_{i=1}^I \int_0^T \int_\Omega \int_{Y^p} \frac{D}{|Y^p|} \left(\nabla_x u_i(t,x) + \sum_{j=1}^n \nabla_y a_j \frac{\partial u_i(t,x,y)}{\partial x_j} \right) \nabla_x \phi_{0_i}(t,x)\, dx\, dy\, dt$$
$$= \sum_{i=1}^I \int_0^T \langle SR(u(t))_i, \phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)}\, dt,$$

i.e.,

$$\sum_{i=1}^I \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_{0_i}(t) \right\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} dt$$
$$+ \sum_{i=1}^I \int_0^T \int_\Omega \sum_{j,k=1}^n \left\{ \frac{D}{|Y^p|} \int_{Y^p} \left(\delta_{jk} + \frac{\partial a_j}{\partial y_k} \right) dy \right\} \frac{\partial u_i(t,x)}{\partial x_j} \frac{\partial \phi_{0_i}(t,x)}{\partial x_k}\, dx\, dt$$
$$= \sum_{i=1}^I \int_0^T \langle SR(u(t))_i, \phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)}\, dt,$$

i.e.,

$$\sum_{i=1}^I \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_{0_i}(t) \right\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} dt + \sum_{i=1}^I \int_0^T \int_\Omega P\, \nabla_x u_i(t,x) \nabla \phi_{0_i}(t,x)\, dx\, dt$$
$$= \sum_{i=1}^I \int_0^T \langle SR(u(t))_i, \phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)}\, dt, \tag{4.1.99}$$

where P is a second order tensor whose components are given as

$$p_{jk} = \int_{Y^p} \frac{D}{|Y^p|}\left(\delta_{jk} + \frac{\partial a_j}{\partial y_k}\right) dy \text{ for all } j,k = 1,2,...,n. \tag{4.1.100}$$

Similarly the boundary condition simplifies to

$$P\,\nabla u \cdot \vec{n} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega. \tag{4.1.101}$$

Therefore the strong form of the complete homogenized problem is

$$\frac{\partial u}{\partial t} - \nabla\cdot P\,\nabla u = SR(u) \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.1.102}$$

$$-P\,\nabla u\cdot\vec{n} = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega, \tag{4.1.103}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \Omega. \tag{4.1.104}$$

Let us denote this problem by (P^1).

Proposition 4.1.2.3.1. *The tensor $P = (p_{jk})_{\substack{1\le j\le n\\ 1\le k\le n}}$ is a second order positive definite symmetric tensor.*

Proof. This follows from the definition of P. For details see in [HJ91] or lemma 5.2 in [Pet03]. ♦

Theorem 4.1.2.3.2. *There exists a unique solution $u \in \mathcal{F}_p^u \cap L^\infty((0,T);L^\infty(\Omega))^I$ of the homogenized problem (4.1.102)-(4.1.104).*

Proof. From (4.1.72) and (4.1.78), it follows that the *two-scale* limit $u \in [H^{1,2}((0,T); H^{1,2}(\Omega)^*)\cap L^2((0,T);H^{1,2}(\Omega))\cap L^\infty((0,T)\times\Omega)]^I$. We still have two things to prove:

- Uniqueness of the solution of (P^1)
- $u\in\mathcal{F}_p^u$

We start by proving the uniqueness of the solution. Let u_1 ad u_2 be the solutions of (4.1.102)-(4.1.104) such that $u_1 \neq u_2$ and $u_1(0,x) = u_2(0,x)$. Proceeding in a similar fashion as in the section 4.1.1.4 we obtain[24]

$$\begin{aligned}\int_0^t \left\langle \frac{\partial}{\partial\theta}(u_{1_i}(\theta)-u_{2_i}(\theta)),(u_{1_i}(\theta)-u_{2_i}(\theta))\right\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)} d\theta \\ +\int_0^t\int_\Omega \nabla(u_{1_i}(\theta,x)-u_{2_i}(\theta,x))P\nabla(u_{1_i}(\theta,x)-u_{2_i}(\theta,x))\,dx\,d\theta \\ = \int_0^t \langle SR(u_1(\theta))_i - SR(u_2(\theta))_i,(u_{1_i}(\theta)-u_{2_i}(\theta))\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,d\theta,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\int_0^t \frac{d}{d\theta}||u_{1_i}(\theta)-u_{2_i}(\theta)||^2_{L^2(\Omega)}\,d\theta + C\int_0^t\int_\Omega ||\nabla(u_{1_i}(\theta,x)-u_{2_i}(\theta,x))||^2\,dx\,d\theta \\ \le \int_0^t\int_\Omega |SR(u_1(\theta,x))_i - SR(u_2(\theta,x))_i|\,|(u_{1_i}(\theta,x)-u_{2_i}(\theta,x))|\,dx\,d\theta\end{aligned}$$

Arguing as in section 4.1.1.4, we get

$$||u_1(t)-u_2(t)||^2_{L^2(\Omega)^I} \le C\int_0^t ||u_1(\theta)-u_2(\theta)||^2_{L^2(\Omega)^I}\,d\theta.$$

[24] Here we have used the positive definiteness of the tensor P.

Gronwall's inequality yields

$$||(u_1(t)-u_2(t))||^2_{L^2(\Omega)^I} = 0 \text{ for a.e. } t$$
$$\Longrightarrow \quad u_1 = u_2,$$

i.e., the solution of (4.1.102)-(4.1.104) is unique.

The abstract formulation of the problem (4.1.102)-(4.1.104) is given by

$$\frac{\partial u(t)}{\partial t} + Au(t) = f(t), \tag{4.1.105}$$
$$u(0,x) = u_0(x), \tag{4.1.106}$$

where $f(t) = SR(u(t)) + \kappa u(t)$, $\kappa > 0$ and the operator $A : H^{1,p}(\Omega)^I \to [H^{1,q}(\Omega)^*]^I$ is defined as in remark 4.1.1.1.1 which has maximal parabolic regularity on $[H^{1,q}(\Omega)^*]^I$. Since $u \in L^\infty((0,T)\times\Omega)^I$, $SR(u)+\kappa u \in L^p((0,T);L^p(\Omega))^I$. The embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$ implies $SR(u)+\kappa u \in L^p((0,T);H^{1,q}(\Omega)^*)^I$. Furthermore, theorem 3.4.2.4 shows that u_0 is in $[(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}]^I$. Therefore by theorem 3.3.1, there exists a unique solution u in $\mathcal{F}^u_p$ of the problem (4.1.105)-(4.1.106) such that

$$|||u|||_{\mathcal{F}^u_p} \le \tilde{C}\left(|||u_0|||_{[(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}]^I} + |||f|||_{L^p((0,T);H^{1,q}(\Omega)^*)^I}\right), \tag{4.1.107}$$

where $\tilde{C} > 0$ depends only on p but is independent of u, u_0 and f. In other words, the problem (P^1) has a unique positive global weak solution u in $\mathcal{F}^u_p$. ♦

4.2 Model M2

4.2.1 Existence and Uniqueness of the Global Solution of (P^2_ε)

Suppose that the following assumptions hold:

(i) $p > n+2$. (4.2.1)

(ii) u_0, v_0 and $w_0 \ge 0$, i.e., u_{0_i}, v_{0_k} and $w_{0_m} \ge 0$ for all $i = 1,2,...,I_1$, $k = 1,2,...,I_2$, and $m = 1,2,...,I_2$. (4.2.2)

(iii) $u_{0_i}, v_{0_k} \in (H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}$ for all $i = 1,2,...,I$, and $k = 1,2,...,I_2$. (4.2.3)

(iv) All the reactions are linearly independent such that the stoichiometric matrices $S_1 = (s_{ij})_{\substack{1\le i\le I_1\\ 1\le j\le J}}$ and $S_2 = (\nu_{ij})_{\substack{1\le k\le I_2\\ 1\le j\le J}}$ has the maximal column rank, i.e.,

$$rank(S_1) = J \text{ and } rank(S_2) = J. \tag{4.2.4}$$

(v)
$$\left.\begin{aligned}&\sup_{\varepsilon>0}||u_{0_i}||_{(H^{1,q}(\Omega^p_\varepsilon)^*,H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} < \infty \text{ for all } i = 1,2,...,I_1,\\ &\sup_{\varepsilon>0}||v_{0_k}||_{(H^{1,q}(\Omega^p_\varepsilon)^*,H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} < \infty \text{ for all } k = 1,2,...,I_2,\\ &\text{and } \sup_{\varepsilon>0}||w_{0_m}||_{L^p(\Omega)} < \infty \text{ for all } m = 1,2,...,I_2.\end{aligned}\right\} \tag{4.2.5}$$

(vi) $\vec{q}_\varepsilon$ is the given fluid velocity which satisfies

$$\nabla\cdot\vec{q}_\varepsilon = 0 \text{ in } \Omega^p_\varepsilon,\ -\vec{q}_\varepsilon\cdot\vec{n} > 0 \text{ on } \partial\Omega_{in},\ -\vec{q}_\varepsilon\cdot\vec{n} \le 0 \text{ on } \partial\Omega_{out} \text{ and } \vec{q}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon. \tag{4.2.6}$$

(vii) $\vec{q}_\varepsilon \in L^\infty((0,T)\times\Omega^p_\varepsilon)$ such that $Q := \sup_{\varepsilon>0}||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega^p_\varepsilon)} < \infty$ and

$$\vec{q}_\varepsilon\cdot\vec{n} \in L^\infty((0,T)\times\partial\Omega_{in}). \tag{4.2.7}$$

$(viii)$ $d_i \le 0$ and $d_i \in L^\infty((0,T)\times\partial\Omega_{in})$ for all $i = 1,2,...I_1$. (4.2.8)

(ix) For a $\tau > 0$, define $v_{\varepsilon_\delta,\tau} := v_{\varepsilon_\delta} + \tau$. We assume that

$$\langle S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta,\tau}), \mu^0 + \log v_{\varepsilon_\delta,\tau}\rangle_{I_2} \leq 0. \tag{4.2.9}$$

(x) For a $\varkappa > 0$, define $u_{\varepsilon_\delta,\varkappa} := u_{\varepsilon_\delta} + \varkappa$. We assume that

$$\langle S_1 R(u_{\varepsilon_\delta,\varkappa}, v_{\varepsilon_\delta}), \bar{\mu}^0 + \log u_{\varepsilon_\delta,\varkappa}\rangle_{I_1} \leq 0. \tag{4.2.10}$$

Remark 4.2.1.1. The suffix δ above is a regularization parameter (see section 4.2.1.1). μ^0 and $\bar{\mu}^0$ are defined in (4.2.53) and (4.2.113) respectively. The assumptions (4.2.9) and (4.2.10) are very strong. The proofs of the inequalities (4.2.9) and (4.2.10) are still open, however, we believe that (4.2.9) and (4.2.10) can be proven.

The assumption (vii) implies

$$\int_0^T \int_{\Omega_\varepsilon^p} |\vec{q}_\varepsilon|^p \, dx\, dt \leq \operatorname*{ess\,sup}_{(0,T)\times\Omega_\varepsilon^p} |\vec{q}_\varepsilon|^p \int_0^T \int_\Omega dx\, dt \leq \sup_{\varepsilon>0} ||\vec{q}_\varepsilon||^p_{L^\infty((0,T)\times\Omega_\varepsilon^p)} \; T\; |\Omega| < \infty$$
$$\implies \quad \sup_{\varepsilon>0} ||\vec{q}_\varepsilon||_{L^p((0,T);L^p(\Omega_\varepsilon^p))} < \infty.$$

Let $\vec{Q}_\varepsilon$ be the extension of $\vec{q}_\varepsilon$ defined as follows:

$$\vec{Q}_\varepsilon := \begin{cases} \vec{q}_\varepsilon & \text{in } (0,T)\times\Omega_\varepsilon^p \\ 0 & \text{in } (0,T)\times\Omega_\varepsilon^s. \end{cases}$$

For the sake of brevity, we still denote the extension of $\vec{q}_\varepsilon$ by $\vec{q}_\varepsilon$. We see that the extended velocity is bounded in $L^p((0,T);L^p(\Omega))$, hence in $L^2((0,T);L^2(\Omega))$. Therefore $\vec{q}_\varepsilon$ is two-scale convergent to the limit $\vec{q}_1$ in $L^2((0,T);L^2(\Omega\times Y))$ and weakly convergent to $\vec{q} = \int_Y \vec{q}_1\, dy$ in $L^2((0,T);L^2(\Omega))$.

4.2.1.1 Regularization of the Function $\psi(w_{\varepsilon_m})$

We can notice that there is a discontinuity in the ODE (2.5.32). Therefore in order to prove the existence of the global weak solution of the problem (2.5.21)-(2.5.35), we introduce a regularization of the function $\psi(w_{\varepsilon_m})$. Let us choose $0 < \delta < 1$ such that $\varepsilon < \delta^p < \delta^2 < 1$. We call this regularized function as $\psi_\delta(w_{\varepsilon_{\delta_m}})$ and is defined by[25]

$$\psi_\delta(w_{\varepsilon_{\delta_m}}) = \begin{cases} 0 & \text{if } w_{\varepsilon_{\delta_m}} \leq 0, \\ \dfrac{w_{\varepsilon_{\delta_m}}}{\delta} & \text{if } 0 < w_{\varepsilon_{\delta_m}} < \delta, \\ 1 & \text{if } w_{\varepsilon_{\delta_m}} \geq \delta. \end{cases} \tag{4.2.11}$$

Our regularized problem is given as:

$$\boxed{\begin{aligned} \frac{\partial u_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta}) &= S_1 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) && \text{in } (0,T)\times\Omega_\varepsilon^p, && (4.2.12)\\ -(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta})\cdot\vec{n} &= d && \text{on } (0,T)\times\partial\Omega_{in}, && (4.2.13)\\ -D\nabla u_{\varepsilon_\delta}\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{out}, && (4.2.14)\\ -D\nabla u_{\varepsilon_\delta}\cdot\vec{n} &= 0 && \text{on } (0,T)\times\Gamma_\varepsilon, && (4.2.15)\\ u_{\varepsilon_\delta}(0,x) &= u_0(x) && \text{in } \Omega_\varepsilon^p, && (4.2.16)\end{aligned}}$$

[25] The function $\psi_\delta(w_{\varepsilon_{\delta_m}})$ is Lipschitz and monotonically increasing on $[0,\delta]$. We sometimes also use the notation $\psi_{\delta_m}(w_{\varepsilon_\delta}) = \psi_\delta(w_{\varepsilon_{\delta_m}})$.

$$\boxed{\begin{array}{lll} \dfrac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) = S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) & \text{in } (0,T)\times\Omega_\varepsilon^p, & (4.2.17)\\ -(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 & \text{on } (0,T)\times\partial\Omega_{in}, & (4.2.18)\\ -D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 & \text{on } (0,T)\times\partial\Omega_{out}, & (4.2.19)\\ -D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\dfrac{\partial w_{\varepsilon_\delta}}{\partial t} & \text{on } (0,T)\times\Gamma_\varepsilon, & (4.2.20)\\ v_{\varepsilon_\delta}(0,x) = v_0(x) & \text{in } \Omega_\varepsilon^p, & (4.2.21)\\ \\ \dfrac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d\psi_\delta(w_{\varepsilon_\delta}) & \text{on } (0,T)\times\Gamma_\varepsilon, & (4.2.22)\\ w_{\varepsilon_\delta}(0,x) = w_0(x) & \text{on } \Gamma_\varepsilon. & (4.2.23) \end{array}}$$

Let us call this problem by $(P^2_{\varepsilon_\delta})$.

Theorem 4.2.1.1.1 (Existence theorem). *Suppose that the assumptions (4.2.1)-(4.2.10) are satisfied, then there exists a unique positive global weak solution* $(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}, w_{\varepsilon_\delta}) \in \mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$ *of the problem*$(P^2_{\varepsilon_\delta})$.

In case of problem $(P^2_{\varepsilon_\delta})$ too[26], we first solve a modified problem. We introduce the rate function $\bar{R} : \mathbb{R}^I \to \mathbb{R}^J$ as

$$\bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) := R(u^+_{\varepsilon_\delta}, v^+_{\varepsilon_\delta}), \tag{4.2.24}$$

where $u^+_{\varepsilon_\delta}$ and $v^+_{\varepsilon_\delta}$ are the positive parts of u_{ε_δ} and v_{ε_δ} respectively, defined componentwise as (4.1.7). Then the problem $(P^2_{\varepsilon_\delta})$ reduces to:

(i) Equations for *type I* species:

$$\boxed{\begin{array}{lll} \dfrac{\partial u_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta}) = S_1 \bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) & \text{in } (0,T)\times\Omega_\varepsilon^p, & (4.2.25)\\ -(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta})\cdot\vec{n} = d & \text{on } (0,T)\times\partial\Omega_{in}, & (4.2.26)\\ -D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 & \text{on } (0,T)\times\partial\Omega_{out}, & (4.2.27)\\ -D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 & \text{on } (0,T)\times\Gamma_\varepsilon, & (4.2.28)\\ u_{\varepsilon_\delta}(0,x) = u_0(x), & \text{in } \Omega_\varepsilon^p, & (4.2.29)\\ \text{where } \bar{d}_i \le 0 \text{ for all } 1 \le i \le I_1. & & (4.2.30) \end{array}}$$

(ii) Equations for *type II* species:

$$\boxed{\begin{array}{lll} \dfrac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) = S_2 \bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) & \text{in } (0,T)\times\Omega_\varepsilon^p, & (4.2.31)\\ -(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 & \text{on } (0,T)\times\partial\Omega_{in}, & (4.2.32)\\ -D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 & \text{on } (0,T)\times\partial\Omega_{out}, & (4.2.33)\\ -D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\dfrac{\partial w_{\varepsilon_\delta}}{\partial t} & \text{on } (0,T)\times\Gamma_\varepsilon, & (4.2.34)\\ v_{\varepsilon_\delta}(0,x) = v_0(x) & \text{in } \Omega_\varepsilon^p. & (4.2.35) \end{array}}$$

(iii) Equations for immobile species:

$$\boxed{\begin{array}{lll} \dfrac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d\psi_\delta(w_{\varepsilon_\delta}) & \text{on } (0,T)\times\Gamma_\varepsilon, & (4.2.36)\\ w_{\varepsilon_\delta}(0,x) = w_0(x) & \text{on } \Gamma_\varepsilon. & (4.2.37) \end{array}}$$

[26] We adopted the idea of [Krä08], [vDP04] and [CHK07].

Let us denote the problem (4.2.25)-(4.2.37) by $(P^{2+}_{\varepsilon\delta})$. We will prove the existence of the global solution of the problem $(P^{2+}_{\varepsilon\delta})$. Since we show that the solution of $(P^{2+}_{\varepsilon\delta})$ is non-negative, it solves the problem $(P^{2}_{\varepsilon\delta})$. We conclude this section by showing the uniqueness of the solution of $(P^{2}_{\varepsilon\delta})$. We commence our investigation with the proof of the positivity of the solution of $(P^{2+}_{\varepsilon\delta})$.

Lemma 4.2.1.1.2. *Let (4.2.1)-(4.2.10) be satisfied. Assume that* $(u_{\varepsilon\delta}, v_{\varepsilon\delta}, w_{\varepsilon\delta}) \in \mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$ *is a solution of the problem* $(P^{2+}_{\varepsilon\delta})$. *Then* $u_{\varepsilon\delta}, v_{\varepsilon\delta}, w_{\varepsilon\delta} \geq 0$ *componentwise, i.e.,* $u_{\varepsilon\delta_i}, v_{\varepsilon\delta_k}$ *and* $w_{\varepsilon\delta_m} \geq 0$ *for all* $i = 1,2,...,I_1, k = 1,2,...,I_2$ *and* $m = 1,2,...,I_2$ *in* $(0,T) \times \Omega^p_\varepsilon$.

Proof. (a) **Positivity of *type I* species:** Let $1 \leq i \leq I_1$. Since $u_{\varepsilon\delta_i}(t) \in H^{1,p}(\Omega^p_\varepsilon)$ for a.e. $0 < t < T$, we have $u^-_{\varepsilon\delta_i}(t) \in H^{1,p}(\Omega^p_\varepsilon)$. Testing the *i-th* PDE of (4.2.25) by $-u^-_{\varepsilon\delta_i}$, we obtain[27]

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} + D\left|\left|\nabla u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} + \int_{\Omega^p_\varepsilon} \vec{q}_\varepsilon \cdot \nabla u^-_{\varepsilon\delta_i}(t) u^-_{\varepsilon\delta_i}(t)\,dx \\
&\quad + \int_{\partial\Omega_{in}} (-d_i u^-_{\varepsilon\delta_i} - \vec{q}_\varepsilon \cdot \vec{n}\left|u^-_{\varepsilon\delta_i}(t)\right|^2)\,ds = -\int_{\Omega^p_\varepsilon} (S_1\bar{R}(u_{\varepsilon\delta}(t), v_{\varepsilon\delta}(t)))_i u^-_{\varepsilon\delta_i}(t)\,dx,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} + D\left|\left|\nabla u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} + \underbrace{\int_{\partial\Omega_{in}} -d_i u^-_{\varepsilon\delta_i}(t)\,ds}_{\geq 0} + \underbrace{\int_{\partial\Omega_{in}} -\vec{q}_\varepsilon \cdot \vec{n}\left|u^-_{\varepsilon\delta_i}(t)\right|^2 ds}_{\geq 0} \\
&\quad \leq \int_{\Omega^p_\varepsilon} |\vec{q}_\varepsilon|\left|u^-_{\varepsilon\delta_i}(t)\right|\left|\nabla u^-_{\varepsilon\delta_i}(t)\right|dx + \int_{\Omega^p_\varepsilon} -S_1 R(u^+_{\varepsilon\delta}(t), v^+_{\varepsilon\delta}(t))_i u^-_{\varepsilon\delta_i}(t)\,dx,^{28}
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} + D\left|\left|\nabla u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} \\
&\quad \leq \frac{Q^2}{2D}\int_{\Omega^p_\varepsilon}\left|u^-_{\varepsilon\delta_i}(t)\right|^2 dx + \frac{D}{2}\int_{\Omega^p_\varepsilon}\left|\nabla u^-_{\varepsilon\delta_i}(t)\right|^2 dx - \int_{\Omega^p_\varepsilon} S_1 R(u^+_{\varepsilon\delta}(t), v^+_{\varepsilon\delta}(t))_i u^-_{\varepsilon\delta_i}(t)\,dx,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} + \frac{D}{2}\left|\left|\nabla u^-_{\varepsilon\delta_i}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)} \\
&\quad \leq \frac{Q^2}{2D}\int_{\Omega^p_\varepsilon}\left|u^-_{\varepsilon\delta_i}(t)\right|^2 dx - \int_{\Omega^p_\varepsilon} S_1 R(u^+_{\varepsilon\delta}(t), v^+_{\varepsilon\delta}(t))_i u^-_{\varepsilon\delta_i}(t)\,dx. \qquad (4.2.38)
\end{aligned}$$

[27]Here we have used the boundary condtions (4.2.26)-(4.2.28) for *type I* species. Since $p > n+2$, from theorem 3.4.3.4 it follows that $u_{\varepsilon\delta} \in \mathcal{F}^u_\varepsilon$ and $v_{\varepsilon\delta} \in \mathcal{G}^v_\varepsilon$ which implies $u_{\varepsilon\delta} \in L^\infty((0,T) \times \Omega^p_\varepsilon)^{I_1}$ and $v_{\varepsilon\delta} \in L^\infty((0,T) \times \Omega^p_\varepsilon)^{I_2}$ respectively. This gives $S_1\bar{R}(u_{\varepsilon\delta}, v_{\varepsilon\delta})_i \in L^p((0,T); L^p(\Omega^p_\varepsilon))$. But $L^p(\Omega^p_\varepsilon) \hookrightarrow H^{1,q}(\Omega^p_\varepsilon)^*$. Thus by the definition (3.1.3) we have

$$\left\langle S_1\bar{R}(u_{\varepsilon\delta}, v_{\varepsilon\delta}), u_{\varepsilon\delta_i}\right\rangle_{H^{1,q}(\Omega^p_\varepsilon)^* \times H^{1,q}(\Omega^p_\varepsilon)} = \left\langle S_1\bar{R}(u_{\varepsilon\delta}, v_{\varepsilon\delta}), u_{\varepsilon\delta_i}\right\rangle_{L^p(\Omega^p_\varepsilon) \times L^q(\Omega^p_\varepsilon)}.$$

Now we simplify the second term on the r.h.s. of (4.2.38).

$$
\begin{aligned}
&-(S_1 R(u^+_{\varepsilon\delta}, v^+_{\varepsilon\delta}))_i \\
&= -\sum_{j=1}^{J} s_{ij} \left(k^f_j \prod_{\substack{m=1 \\ s_{mj}<0}}^{I_1} (u^+_{\varepsilon\delta_m})^{-s_{mj}} \prod_{\substack{m=1 \\ \nu_{mj}<0}}^{I_2} (v^+_{\varepsilon\delta_m})^{-\nu_{mj}} - k^b_j \prod_{\substack{m=1 \\ s_{mj}>0}}^{I_1} (u^+_{\varepsilon\delta_m})^{s_{mj}} \prod_{\substack{m=1 \\ \nu_{mj}>0}}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu_{mj}} \right) \\
&= -\sum_{j=1}^{J} \left(s^+_{ij} k^f_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^-_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^-_{mj}} + s^-_{ij} k^b_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^+_{mj}} \right) \\
&\qquad + \sum_{j=1}^{J} \left(s^-_{ij} k^f_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^-_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^-_{mj}} + s^+_{ij} k^b_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^+_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^+_{mj}} \right) \\
&= Term\,1 + Term\,2,
\end{aligned}
\tag{4.2.39}
$$

where

$$
Term\,1 = -\sum_{j=1}^{J} \left(s^+_{ij} k^f_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^-_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^-_{mj}} + s^-_{ij} k^b_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^+_{mj}} \right)
\tag{4.2.40}
$$

and

$$
Term\,2 = \sum_{j=1}^{J} \left(s^-_{ij} k^f_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^-_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^-_{mj}} + s^+_{ij} k^b_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^+_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^+_{mj}} \right).
\tag{4.2.41}
$$

Since always either $s^-_{ij} = 0$, or $s^+_{ij} = 0$ holds, therefore for $m = i$ in (4.2.39) we see that $Term\,1$ is independent of $u^+_{\varepsilon\delta_i}$ but contains the product of higher powers of $u^+_{\varepsilon\delta_l}$ and $\prod_{m=1}^{I_2}(v^+_{\varepsilon\delta})^{\nu^-_{mj}}$ for all $l = 1, 2, ..., I_1, l \neq i$ whereas $Term\,2$ contains a factor $(u^+_{\varepsilon\delta_i})^r$ with $r = s^-_{ij}$ or $s^+_{ij} \geq 1$ and the product of higher powers of $u^+_{\varepsilon\delta_l}$ and $\prod_{m=1}^{I_2}(v^+_{\varepsilon\delta})^{\nu^+_{mj}}$ for all $l = 1, 2, ..., I_1, l \neq i$. Since for all l and k, $u^+_{\varepsilon\delta_l}, v^+_{\varepsilon\delta_k} \geq 0$, this gives

$$
\begin{aligned}
&(S_1 R(u^+_{\varepsilon\delta}, v^+_{\varepsilon\delta}))_i \\
&\leq \sum_{j=1}^{J} \left(s^-_{ij} k^f_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^-_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^-_{mj}} + s^+_{ij} k^b_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^+_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^+_{mj}} \right).
\end{aligned}
$$

Moreover $u^+_{\varepsilon\delta_i}(t) = 0$ in the support of $u^-_{\varepsilon\delta_i}(t)$ and $u^-_{\varepsilon\delta_i}(t) = 0$ in the support of $u^+_{\varepsilon\delta_i}(t)$, therefore

$$
\begin{aligned}
&\int_{\Omega^p_\varepsilon} -S_1 R(u^+_{\varepsilon\delta}(t), v^+_{\varepsilon\delta}(t)) u^-_{\varepsilon\delta_i}(t)\, dx \\
&\leq \int_{\Omega^p_\varepsilon} \sum_{j=1}^{J} \left(s^-_{ij} k^f_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^-_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^-_{mj}} + s^+_{ij} k^b_j \prod_{m=1}^{I_1} (u^+_{\varepsilon\delta_m})^{s^+_{mj}} \prod_{m=1}^{I_2} (v^+_{\varepsilon\delta_m})^{\nu^+_{mj}} \right) u^-_{\varepsilon\delta_i}\, dx \\
&= 0.
\end{aligned}
\tag{4.2.42}
$$

From (4.2.38) and (4.2.42) we get

$$
\frac{d}{dt} \left\| u^-_{\varepsilon\delta_i}(t) \right\|^2_{L^2(\Omega^p_\varepsilon)} \leq \frac{Q^2}{D} \left\| u^-_{\varepsilon\delta_i}(t) \right\|^2_{L^2(\Omega^p_\varepsilon)}.
\tag{4.2.43}
$$

[28] From (4.2.8), $-d_i \geq 0$; from (4.2.6), $-\vec{q}_\varepsilon \cdot \vec{n} \geq 0$; and by definition (4.1.7), $u^-_{\varepsilon\delta_i} \geq 0$.

Note that $u_{\varepsilon_{\delta_i}}(0) > 0$, i.e, $u^-_{\varepsilon_{\delta_i}}(0) = 0$. A straightforward application of Gronwall's inequality gives

$$\begin{aligned}
\left\|u^-_{\varepsilon_{\delta_i}}(t)\right\|^2_{L^2(\Omega^p_\varepsilon)} &= 0 \quad \text{for a.e. } t \text{ and for all } i = 1,2,\ldots,I_1,\\
\Longrightarrow\ u^-_{\varepsilon_{\delta_i}} &= 0 \quad \text{for a.e. in } (0,T)\times\Omega^p_\varepsilon \text{ and for all } i = 1,2,\ldots,I_1,\\
\Longrightarrow\ u_{\varepsilon_{\delta_i}} &\geq 0 \quad \text{for a.e. in } (0,T)\times\Omega^p_\varepsilon \text{ and for all } i = 1,2,\ldots,I_1.
\end{aligned}$$

(b) **Positivity of the *type-II* species:** In this case testing the k-*th* PDE of (4.2.31) by $-v^-_{\varepsilon_{\delta_k}}$ and proceeding in the same way as in part (a) yields the proof.
(c) **Positivity of the solution for the immobile species:** The positivity follows by similar arguments as in lemma 3.1 in [Krä08]. ♦

4.2.1.2 Existence of the Global Solution of the Problem (4.2.36)-(4.2.37)

Theorem 4.2.1.2.1. *Let $(\hat{u}_{\varepsilon_\delta}, \hat{v}_{\varepsilon_\delta}) \in \mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon$. Then there exists a positive global solution $w_{\varepsilon_\delta} \in \mathcal{H}^w_\varepsilon$ of the problem (4.2.36)-(4.2.37).*

Proof. (i) **Positivity:** This is shown in the lemma 4.2.1.1.2.
(ii) **Existence of the local solution:** Let $x \in \Omega^p_\varepsilon$ be fixed but chosen arbitrarily. The function $\psi_{\delta_m}(.)$ is continuous w.r.t. $w_{\varepsilon_{\delta_m}}$ for every fixed t. Since

$$|\psi_\delta(w_{\varepsilon_\delta}) - \psi_\delta(\bar{w}_{\varepsilon_\delta})|^2_I = \sum_{m=1}^{I_2}\left|\psi_\delta(w_{\varepsilon_{\delta_m}}) - \psi_\delta(\bar{w}_{\varepsilon_{\delta_m}})\right|^2,$$

the vector function $\psi_\delta(.)$ is also continuous w.r.t. w_{ε_δ} for every fixed t. Moreover, $\psi_\delta(w_{\varepsilon_\delta})$ is measurable w.r.t. t and

$$|\psi_\delta(w_{\varepsilon_\delta})|_I = \left[\sum_{m=1}^{I_2}\left|\psi_\delta(w_{\varepsilon_{\delta_m}})\right|^2\right]^{\frac{1}{2}} \leq \left[\sum_{m=1}^{I_2} 1\right]^{\frac{1}{2}} = I_2^{\frac{1}{2}} =: m(t),$$

i.e., $\psi_\delta(.)$ is bounded by a measurable function $m(.)$. Thus the application of the *Caratheodory's theorem* yields the existence of an absolutely continuous function $w_{\varepsilon_\delta}(x)$ on $[0,T_1)$ which solves (4.2.36)-(4.2.37) (cf. theorem 2.1.1 in [CL55]), i.e., $w_{\varepsilon_\delta}(x) \in [H^{1,1}(0,T_1)]^{I_2}$, where $T_1 \leq T$, i.e, the solution is local. Since x is arbitrary, for a.e. $x \in \Omega^p_\varepsilon$, $w_{\varepsilon_\delta}(x) \in [H^{1,1}(0,T_1)]^{I_2}$. For all $\phi \in [C_0^\infty((0,T_1))]^{I_2}$, the weak formulation of (4.2.36) is given by

$$\int_0^{T_1}\left\langle \frac{\partial w_{\varepsilon_\delta}(t)}{\partial t}, \phi(t)\right\rangle_{I_2} dt = -k_d\int_0^{T_1}\langle\psi_\delta(w_{\varepsilon_\delta}(t)),\phi(t)\rangle_{I_2}\,dt,$$

i.e.,

$$\int_0^{T_1}\left\langle w_{\varepsilon_\delta}(t), \frac{\partial\phi(t)}{\partial t}\right\rangle_{I_2} dt = k_d\int_0^{T_1}\langle\psi_\delta(w_{\varepsilon_\delta}(t)),\phi(t)\rangle_{I_2}\,dt. \tag{4.2.44}$$

Since w_{ε_δ} is a function of both x and t, we shall show that the weak derivative of w_{ε_δ} depends on both x and t and belongs to $L^p((0,T);L^p(\Gamma_\varepsilon))^{I_2}$ which accomplishes the claim that $w_{\varepsilon_\delta} \in \mathcal{H}^w_\varepsilon$. Let us choose another function $\zeta \in C_0^\infty(\Gamma_\varepsilon)$. Multiplying (4.2.44) by ζ and integrating over Γ_ε, we obtain

$$\int_0^{T_1}\int_{\Gamma_\varepsilon}\left\langle w_{\varepsilon_\delta}(t,x), \frac{\partial\phi(t)}{\partial t}\zeta(x)\right\rangle_{I_2} ds\,dt = k_d\int_0^{T_1}\int_{\Gamma_\varepsilon}\langle\psi_\delta(w_{\varepsilon_\delta}(t,x)),\phi(t)\zeta(x)\rangle_{I_2}\,ds\,dt, \tag{4.2.45}$$

for all $\phi \in [C_0^\infty((0,T_1))]^{I_2}$ and $\zeta \in C_0^\infty(\Gamma_\varepsilon)$. As $\phi \in [C_0^\infty((0,T_1))]^{I_2}$ and $\zeta \in C_0^\infty(\Gamma_\varepsilon)$, $\phi\zeta \in L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$ such that the weak time derivative is in $L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$, i.e.,

$$\int_0^{T_1} \left\langle w_{\varepsilon_\delta}, \frac{\partial \phi(t)}{\partial t}\psi \right\rangle_{L^p(\Gamma_\varepsilon)^{I_2}\times L^q(\Gamma_\varepsilon)^{I_2}} dt = k_d \int_0^{T_1} \langle \psi_\delta(w_{\varepsilon_\delta}(t)), \phi(t)\psi\rangle_{L^p(\Gamma_\varepsilon)^{I_2}\times L^q(\Gamma_\varepsilon)^{I_2}}\, dt.$$

Therefore for any $\eta \in L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$ such that $\frac{\partial \eta}{\partial t} \in L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$, we have

$$\int_0^{T_1} \left\langle w_{\varepsilon_\delta}(t,x), \frac{\partial \eta(t,x)}{\partial t} \right\rangle_{L^p(\Gamma_\varepsilon)^{I_2}\times L^q(\Gamma_\varepsilon)^{I_2}} dt = k_d \int_0^{T_1} \langle \psi_\delta(w_{\varepsilon_\delta}(t,x)), \eta(t,x)\rangle_{L^p(\Gamma_\varepsilon)^{I_2}\times L^q(\Gamma_\varepsilon)^{I_2}}\, dt.$$

This leads to the fact that the weak derivative of w_{ε_δ}, $\frac{\partial w_{\varepsilon_\delta}}{\partial t} \in L^p((0,T_1);L^p(\Gamma_\varepsilon))^{I_2}$, i.e., $w_{\varepsilon_\delta} \in H^{1,p}((0,T_1);L^p(\Gamma_\varepsilon))^{I_2}$.
(iii) **Extension of the solution:** Clearly,

$$|\psi_\delta(w_{\varepsilon_\delta})|_I = \left[\sum_{k=1}^{I_2} \left|\psi_\delta(w_{\varepsilon_{\delta_i}})\right|^2\right]^{\frac{1}{2}} \le \left[\sum_{i=1}^{I_2} 1^2\right]^{\frac{1}{2}} = I_2^{\frac{1}{2}},$$

i.e., the r.h.s. of (4.2.36)-(4.2.37) is bounded. Therefore from corollary II.3.4 of [MM02], there exists a global solution of the problem (4.2.36)-(4.2.37) on $[0,T)$ for any $T > 0$.

♦

4.2.1.3 Existence of the Global Solution of the Problem (4.2.31)-(4.2.37)

Lemma 4.2.1.3.1. *Suppose that $p > n+2$ is fixed and $\varkappa \in L^\infty(\partial\Omega_{in})$. If we define the map $Q_{\partial\Omega_{in}} : [H^{1,p}(\Omega_\varepsilon^p)]^{I_2} \to [H^{1,q}(\Omega_\varepsilon^p)]^{I_2 *}$ by*

$$\langle Q_{\partial\Omega_{in}}(\phi), \xi\rangle := \sum_{k=1}^{I_2} \langle Q_{\partial\Omega_{in}}(\phi_k), \xi_k\rangle := \sum_{k=1}^{I_2} \int_{\partial\Omega_{in}} \varkappa \phi_k \xi_k \, ds \quad \textit{for } \xi \in [H^{1,q}(\Omega_\varepsilon^p)]^{I_2}, \tag{4.2.46}$$

then $Q_{\partial\Omega_{in}}$ is well defined and continuous.

Proof. For $\phi \in H^{1,p}(\Omega_\varepsilon^p)^{I_2}$, the map is given by

$$\begin{aligned}
\langle Q_{\partial\Omega_{in}}(\phi), \xi\rangle &= \sum_{k=1}^{I_2} \int_{\partial\Omega_{in}} \varkappa \phi_k \xi_k \, ds \\
&\le ||\varkappa||_{L^\infty(\partial\Omega_{in})} \sum_{k=1}^{I_2} \int_{\partial\Omega_{in}} |\phi_k|\,|\xi_k| \, ds \\
&\le ||\varkappa||_{L^\infty(\partial\Omega_{in})} \sum_{k=1}^{I_2} \int_{\partial\Omega} |\phi_k|\,|\xi_k| \, ds \\
&\le ||\varkappa||_{L^\infty(\partial\Omega_{in})} \sum_{k=1}^{I_2} ||\phi_k||_{L^p(\partial\Omega)}\, ||\xi_k||_{L^q(\partial\Omega)}\,.
\end{aligned} \tag{4.2.47}$$

From theorem 3.4.2.2, we know that for $1 \le p < \infty$ and $\phi_k \in H^{1,p}(\Omega_\varepsilon^p)$, there exists an extension $\tilde{\phi}_k$ of ϕ_k (for the sake of notation we still denote the extension by ϕ_k) such that

$$||\phi_k||_{H^{1,p}(\Omega)} \le C\, ||\phi_k||_{H^{1,p}(\Omega_\varepsilon^p)}\,, \tag{4.2.48}$$

where C is independent of ε. Also from theorem B.5, for a domain Ω with sufficiently smooth boundary and for $1 \le p < \infty$, there exists a bounded linear operator $T : H^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that for $\phi_k \in H^{1,p}(\Omega_\varepsilon^p)$, $T\phi_k := \phi_k|_{\partial\Omega}$ and

$$||\phi_k||_{L^p(\partial\Omega)} \le C\, ||\phi_k||_{H^{1,p}(\Omega)}\,, \tag{4.2.49}$$

where C depends on p and Ω only. Combining (4.2.48) and (4.2.49), we obtain

$$||\phi_k||_{L^p(\partial\Omega)} \le C\,||\phi_k||_{H^{1,p}(\Omega)} \le C\,||\phi_k||_{H^{1,p}(\Omega_\varepsilon^p)}. \tag{4.2.50}$$

An inequality similar to (4.2.50) holds for ξ_k too, i.e.,

$$||\xi_k||_{L^q(\partial\Omega)} \le C\,||\xi_k||_{H^{1,q}(\Omega)} \le C\,||\xi_k||_{H^{1,q}(\Omega_\varepsilon^p)}. \tag{4.2.51}$$

Using (4.2.50) and (4.2.51) in (4.2.47), we get

$$\begin{aligned}
&|\langle Q_{\partial\Omega_{in}}(\phi),\xi\rangle| \\
&\le C\sum_{k=1}^{I_2} ||\phi_k||_{H^{1,p}(\Omega_\varepsilon^p)}\,||\xi_k||_{H^{1,q}(\Omega_\varepsilon^p)} \\
&\le C\left[\sum_{k=1}^{I_2} ||\phi_k||^p_{H^{1,p}(\Omega_\varepsilon^p)}\right]^{\frac{1}{p}}\left[\sum_{k=1}^{I_2} ||\xi_k||^q_{H^{1,q}(\Omega_\varepsilon^p)}\right]^{\frac{1}{q}}, \text{ by discrete Hölder's inequality} \\
&= C\,|||\phi|||_{[H^{1,p}(\Omega_\varepsilon^p)]^{I_2}}\,|||\xi|||_{[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}} \\
&\Longrightarrow \sup_{|||\xi|||_{[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}}=1} |\langle Q_{\partial\Omega_{in}}(\phi),\xi\rangle| \le C\,|||\phi|||_{[H^{1,p}(\Omega_\varepsilon^p)]^{I_2}}\left(\sup_{|||\xi|||_{[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}}=1} |||\xi|||_{[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}}\right) \\
&\Longrightarrow |||Q_{\partial\Omega_{in}}(\phi)|||_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}} \le C\,|||\phi|||_{[H^{1,p}(\Omega_\varepsilon^p)]^{I_2}} \\
&\Longrightarrow ||Q_{\partial\Omega_{in}}||||_{\mathcal{L}([H^{1,p}(\Omega_\varepsilon^p)]^{I_2},[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2})} \le C.
\end{aligned}$$

This shows that the map $Q_{\partial\Omega_{in}} : [H^{1,p}(\Omega_\varepsilon^p)]^{I_2} \to [H^{1,q}(\Omega_\varepsilon^p)]^{I_2 *}$ is well-defined and bounded, hence continuous. ♦

Lemma 4.2.1.3.2. *Let $p > n+2$ be fixed. Then the map $R_{\Gamma_\varepsilon} : [L^p(\Gamma_\varepsilon)]^{I_2} \to [H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}$ given by*

$$\langle R_{\Gamma_\varepsilon}(\upsilon),\eta\rangle := \sum_{k=1}^{I_2}\langle R_{\Gamma_\varepsilon}(\upsilon_k),\eta_k\rangle := \sum_{k=1}^{I_2}\varepsilon\int_{\Gamma_\varepsilon}\upsilon_k(x)\eta_k(x)\,d\sigma_x,\ \textit{for}\ \eta\in[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}, \tag{4.2.52}$$

is well defined and continuous.

Proof. We proceed like previous lemma. Here the map is given as[29]

$$\begin{aligned}
\langle R_{\Gamma_\varepsilon}(\upsilon),\eta\rangle &= \sum_{k=1}^{I_2}\varepsilon\int_{\Gamma_\varepsilon}\upsilon_k(x)\eta_k(x)\,d\sigma_x \quad \eta\in[H^{1,q}(\Omega_\varepsilon^p)]^{I_2} \\
&\le \sum_{k=1}^{I_2}\varepsilon\left(\int_{\Gamma_\varepsilon}|\upsilon_k(x)|^p\,d\sigma_x\right)^{\frac{1}{p}}\left(\int_{\Gamma_\varepsilon}|\eta_k(x)|^q\,d\sigma_x\right)^{\frac{1}{q}} \\
&\le \sum_{k=1}^{I_2}\left(\varepsilon\int_{\Gamma_\varepsilon}|\upsilon_k(x)|^p\,d\sigma_x\right)^{\frac{1}{p}}\left(\varepsilon\int_{\Gamma_\varepsilon}|\eta_k(x)|^q\,d\sigma_x\right)^{\frac{1}{q}} \\
&\le C\sum_{k=1}^{I_2}||\upsilon_k||_{L^p(\Gamma_\varepsilon)}\left[\int_{\Omega_\varepsilon^p}|\eta_k(x)|^q+\varepsilon^q\,|\nabla\eta_k(x)|^q\right]^{\frac{1}{q}} \\
&\le C\,\max(1,\varepsilon^q)^{\frac{1}{q}}\sum_{k=1}^{I_2}||\upsilon_k||_{L^p(\Gamma_\varepsilon)}\,||\eta_k||_{H^{1,q}(\Omega_\varepsilon^p)}
\end{aligned}$$

[29]Note that we have used the theorem 3.4.1.3. Also see that $\varepsilon = \varepsilon^{\frac{1}{p}+\frac{1}{q}}$.

$$\leq C\ \max(1,\varepsilon^q)^{\frac{1}{q}} \left[\sum_{k=1}^{I_2} ||v_k||^p_{L^p(\Gamma_\varepsilon)}\right]^{\frac{1}{p}} \left[\sum_{k=1}^{I_2} ||\eta_k||^q_{H^{1,q}(\Omega^p_\varepsilon)}\right]^{\frac{1}{q}}$$
$$\leq C\ \max(1,\varepsilon^q)^{\frac{1}{q}}\ |||v|||_{L^p(\Gamma_\varepsilon)^{I_2}}\, |||\eta|||_{H^{1,q}(\Omega^p_\varepsilon)^{I_2}}\,.$$

The proof follows. ♦

Theorem 4.2.1.3.3. *Let the assumptions (4.2.1)-(4.2.10) hold true and $\hat{u}_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$. Then there exists a global weak solution $(v_{\varepsilon_\delta}, w_{\varepsilon_\delta}) \in \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$ of the problem*

$$\begin{aligned}
\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) &= S_2\bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}) && \text{in} \quad (0,T)\times\Omega^p_\varepsilon,\\
-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} &= 0 && \text{on} \quad (0,T)\times\partial\Omega_{in},\\
-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} &= 0 && \text{on} \quad (0,T)\times\partial\Omega_{out},\\
-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} &= \varepsilon\frac{\partial w_{\varepsilon_\delta}}{\partial t} && \text{on} \quad (0,T)\times\Gamma_\varepsilon,\\
v_{\varepsilon_\delta}(0,x) &= v_0(x) && \text{in} \quad \Omega^p_\varepsilon,\\
\frac{\partial w_{\varepsilon_\delta}}{\partial t} &= -k_d\psi_\delta(w_{\varepsilon_\delta}) && \text{on} \quad (0,T)\times\Gamma_\varepsilon,\\
w_{\varepsilon_\delta}(0,x) &= w_0(x) && \text{on} \quad \Gamma_\varepsilon.
\end{aligned}$$

Let us denote this problem by $(\bar{P}^{2+}_{\varepsilon_\delta})$. We follow the approach shown in section 4.1.1 but here we pay special attention to the boundary terms due to the presence of *inflow-outflow boundary* conditions. We define the Lyapunov functional in the following way: Let $\mu^0 \in \mathbb{R}^{I_2}$ be a solution of the linear system

$$S_2^T\mu^0 = -\log K, \tag{4.2.53}$$

where $K \in \mathbb{R}^J$ is the vector of equilibrium constants $K_j = \frac{k^f_j}{k^b_j}$. Due to (4.2.4), the system (4.2.53) has a solution. Let $g_k : \mathbb{R}^+_0 \to \mathbb{R}$ be defined as

$$g_k(v_{\varepsilon_{\delta_k}}) := \left(\mu^0_k - 1 + \log v_{\varepsilon_{\delta_k}}\right) v_{\varepsilon_{\delta_k}} + e^{(1-\mu^0_k)} \qquad \text{for each } k = 1,2,...,I_2. \tag{4.2.54}$$

We define $g : \mathbb{R}^{+^{I_2}}_0 \to \mathbb{R}$, $f_r : \mathbb{R}^{+^{I_2}}_0 \to \mathbb{R}$ and $F_r : L^\infty_+(\Omega^p_\varepsilon)^{I_2} \to \mathbb{R}$ in a similar way as we did in section 4.1.1.2. We also note that all the properties of g_k, g, f_r and F_r from section 4.1.1.2 (see propositions 4.1.1.2.1 and 4.1.1.2.2) hold good. For technical reason, we add an extra term on both sides of the first PDE in the problem $(\bar{P}^{2+}_{\varepsilon_\delta})$, i.e., for any $\kappa > 0$, we have

$$\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) + \kappa v_{\varepsilon_\delta} = S_2\bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}) + \kappa v_{\varepsilon_\delta} \quad \text{in} \quad (0,T)\times\Omega^p_\varepsilon, \tag{4.2.55}$$
$$-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega_{in}, \tag{4.2.56}$$
$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega_{out}, \tag{4.2.57}$$
$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\frac{\partial w_{\varepsilon_\delta}}{\partial t} \quad \text{on} \quad (0,T)\times\Gamma_\varepsilon, \tag{4.2.58}$$
$$v_{\varepsilon_\delta}(0,x) = v_0(x) \quad \text{in} \quad \Omega^p_\varepsilon, \tag{4.2.59}$$
$$\frac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d\psi_\delta(w_{\varepsilon_\delta}) \quad \text{on} \quad (0,T)\times\Gamma_\varepsilon, \tag{4.2.60}$$
$$w_{\varepsilon_\delta}(0,x) = w_0(x) \quad \text{on} \quad \Gamma_\varepsilon. \tag{4.2.61}$$

We denote the problem (4.2.55)-(4.2.61) by $(\bar{P}^{2+}_{\varepsilon_{\delta_M}})$. Since a solution of $(\bar{P}^{2+}_{\varepsilon_{\delta_M}})$ is also a solution of $(\bar{P}^{2+}_{\varepsilon_\delta})$, we prove the global existence of the weak solution of $(\bar{P}^{2+}_{\varepsilon_{\delta_M}})$. Let us define the fixed point operator $Z_1 : \mathcal{G}_\varepsilon^v \to \mathcal{G}_\varepsilon^v$ via $v_{\varepsilon_\delta} := Z_1(\hat{v}_{\varepsilon_\delta})$, where v_{ε_δ} is the solution of the linear problem given by

$$\begin{aligned}
\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla \cdot (D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) + \kappa v_{\varepsilon_\delta} &= S_2 \bar{R}(\hat{u}_{\varepsilon_\delta}, \hat{v}_{\varepsilon_\delta}) + \kappa \hat{v}_{\varepsilon_\delta} && \text{in} \quad (0,T) \times \Omega_\varepsilon^p, && (4.2.62)\\
-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) \cdot \vec{n} &= 0 && \text{on} \quad (0,T) \times \partial\Omega_{in}, && (4.2.63)\\
-D\nabla v_{\varepsilon_\delta} \cdot \vec{n} &= 0 && \text{on} \quad (0,T) \times \partial\Omega_{out}, && (4.2.64)\\
-D\nabla v_{\varepsilon_\delta} \cdot \vec{n} &= \varepsilon \frac{\partial w_{\varepsilon_\delta}}{\partial t} && \text{on} \quad (0,T) \times \Gamma_\varepsilon, && (4.2.65)\\
v_{\varepsilon_\delta}(0,x) &= v_0(x) && \text{in} \quad \Omega_\varepsilon^p, && (4.2.66)\\
\frac{\partial w_{\varepsilon_\delta}}{\partial t} &= -k_d \psi_\delta(w_{\varepsilon_\delta}) && \text{on} \quad (0,T) \times \Gamma_\varepsilon, && (4.2.67)\\
w_{\varepsilon_\delta}(0,x) &= w_0(x) && \text{on} \quad \Gamma_\varepsilon. && (4.2.68)
\end{aligned}$$

Remark 4.2.1.3.4. For fixed $\hat{u}_{\varepsilon_\delta}$ and $\hat{v}_{\varepsilon_\delta}$, the subproblem (4.2.67)-(4.2.68) has a unique positive global solution w_{ε_δ} in $\mathcal{H}_\varepsilon^w$. The reformulation of the problem (4.2.62)-(4.2.66) is given by

$$\begin{aligned}
\frac{\partial v_{\varepsilon_\delta}}{\partial t} + A v_{\varepsilon_\delta} &= f_{bound}(v_{\varepsilon_\delta}) + f(\hat{u}_{\varepsilon_\delta}, \hat{v}_{\varepsilon_\delta}),\\
v_{\varepsilon_\delta}(0,x) &= v_0(x),
\end{aligned} \tag{AP}$$

where A is defined as in the remark 4.1.1.1.1 and satisfies the maximal regularity on $[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}$, $f_{bound}(v_{\varepsilon_\delta}) := Q_{\partial\Omega_{in}}(v_{\varepsilon_\delta}) + R_{\Gamma_\varepsilon}\left(-\frac{\partial w_{\varepsilon_\delta}}{\partial t}\right) - \vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_\delta}$, and $f(\hat{u}_{\varepsilon_\delta}, \hat{v}_{\varepsilon_\delta}) := \kappa \hat{v}_{\varepsilon_\delta} + S_2 \bar{R}(\hat{u}_{\varepsilon_\delta}, \hat{v}_{\varepsilon_\delta})$, where $\kappa > 0$. Note that the theorem 3.4.3.4 implies $\hat{u}_{\varepsilon_\delta} \in L^\infty((0.T) \times \Omega_\varepsilon^p)^{I_1}$. Similar arguments as in remark 4.1.1.1.1 leads to the fact that $f \in L^p((0,T); H^{1,q}(\Omega_\varepsilon^p)^*)^{I_2}$. Using lemmas 4.2.1.3.1, 4.2.1.3.2 and the assumption (4.2.7), the boundary term $f_{bound} \in L^p((0,T); H^{1,q}(\Omega_\varepsilon^p)^*)^{I_2}$. The condition $v_0 \in \left[(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}\right]^{I_2}$ is fulfilled by (4.2.3). Then theorem 3.3.1 assures the existence of a unique solution of the problem (AP). Therefore the operator Z_1 is well-defined.

The application of Schaefer's fixed point theorem resides on the verification of the following two conditions:

(i) The operator Z_1 is continuous and compact.

(ii) The set $\{v_{\varepsilon_\delta} \in \mathcal{G}_\varepsilon^v | \exists \lambda \in [0,1] : v_{\varepsilon_\delta} = \lambda Z_1(v_{\varepsilon_\delta})\}$ is bounded, i.e., there exists a constant $C > 0$ independent of v_{ε_δ} and λ such that any arbitrary solution $v_{\varepsilon_\delta} \in \mathcal{G}_\varepsilon^v$ of the equation

$$v_{\varepsilon_\delta} = \lambda Z_1(v_{\varepsilon_\delta}) \tag{4.2.69}$$

satisfies

$$|||v_{\varepsilon_\delta}|||_{\mathcal{G}_\varepsilon^v} \leq C. \tag{4.2.70}$$

Equations (4.2.62)-(4.2.68) and (4.1.69) imply

$$\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) + \kappa v_{\varepsilon_\delta} = \lambda S_2 \bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}) + \lambda\kappa v_{\varepsilon_\delta} \quad \text{in } (0,T)\times\Omega_\varepsilon^p, \tag{4.2.71}$$

$$v_{\varepsilon_\delta}(0,x) = \lambda v_0(x) \quad \text{in } \Omega_\varepsilon^p, \tag{4.2.72}$$

$$-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.73}$$

$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.74}$$

$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \lambda\varepsilon\frac{\partial w_{\varepsilon_\delta}}{\partial t} \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.75}$$

$$\frac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d\psi_\delta(w_{\varepsilon_\delta}) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.76}$$

$$w_{\varepsilon_\delta}(0,x) = w_0(x) \quad \text{on } \Gamma_\varepsilon. \tag{4.2.77}$$

Note that w_{ε_δ} is the solution of the ODE problem (4.2.76)-(4.2.77). Let us call the problem (4.2.71)-(4.2.77) as $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$. To show the inequality (4.2.70), we aim to prove a theorem like 4.1.1.2.3 which is the following:

Theorem 4.2.1.3.5. *Let $r\in\mathbb{N}$ $(r\geq 2)$, $0\leq t\leq T$ and $0\leq\lambda\leq 1$. Suppose that $\hat{u}_{\varepsilon_\delta}\in\mathcal{F}_\varepsilon^u$. Further assume that $v_{\varepsilon_\delta}\in\mathcal{G}_\varepsilon^v$ is a solution of $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$. Then the following inequality holds good:*

$$F_r(v_{\varepsilon_\delta}(t)) \leq e^{C_{34}t}F_r(v_{\varepsilon_\delta}(0)) \qquad \text{for a.e. } t, \tag{4.2.78}$$

where C_{34} is independent of ε, δ, λ and t.

Our starting point is the following lemma which is similar to the lemma 4.1.1.2.7.

Lemma 4.2.1.3.6. *Let $p>n+2$, $\hat{u}_{\varepsilon_\delta}\in\mathcal{F}_\varepsilon^u$ and $r\in\mathbb{N}$ $(r\geq 2)$. Assume that $v_{\varepsilon_\delta}\in\mathcal{G}_\varepsilon^v$ is a solution of $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$ and for $\tau>0$,*

$$v_{\varepsilon_\delta,\tau} := v_{\varepsilon_\delta} + \tau. \tag{4.2.79}$$

Then the following inequality holds:

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial v_{\varepsilon_\delta,\tau}}{\partial\theta}, \partial f_r(v_{\varepsilon_\delta,\tau})\right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}\times H^{1,q}(\Omega_\varepsilon^p)^{I_2}} d\theta \\ &\leq h(t,\tau,v_{\varepsilon_\delta,\tau}) + l(t,\tau,v_{\varepsilon_\delta,\tau}) + C_{34}\int_0^t F_r(v_{\varepsilon_\delta,\tau})\,d\theta \qquad \text{for a.e. } t,\end{aligned} \tag{4.2.80}$$

where $h(t,\tau,v_{\varepsilon_\delta,\tau})$ and $l(t,\tau,v_{\varepsilon_\delta,\tau})$ tend to zero as $\tau\to 0$ for a.e. t, and C_{34} is independent of ε, δ, λ and t.

Proof. Obviously $v_{\varepsilon_\delta,\tau}\in\mathcal{G}_\varepsilon^v$. For $p>n+2$, $v_{\varepsilon_\delta,\tau}\in L^\infty((0,T)\times\Omega_\varepsilon^p)^{I_2}$ (cf. theorem 3.4.3.4) and $\partial f_r(v_{\varepsilon_\delta,\tau})\in L^q((0,T);H^{1,q}(\Omega_\varepsilon^p))^{I_2}$. Using $\partial f_r(v_{\varepsilon_\delta,\tau})$ in the weak formualtion of the PDE (4.2.71), we get

$$\begin{aligned}&\int_0^t \langle\partial_\theta v_{\varepsilon_\delta}, \partial f_r(v_{\varepsilon_\delta,\tau})\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}\times[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}}\,d\theta + \kappa\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} v_{\varepsilon_{\delta_k}}\partial f_r(v_{\varepsilon_\delta,\tau})_k\,dx\,d\theta \\ &-\sum_{k=1}^{I_2}\left[\int_0^t\int_{\partial\Omega_{in}}\vec{q}_\varepsilon\cdot\vec{n}v_{\varepsilon_{\delta_k}}(\partial f_r(v_{\varepsilon_\delta,\tau}))_k\,ds\,d\theta + \lambda\varepsilon k_d\int_0^t\int_{\Gamma_\varepsilon}\psi_\delta(w_{\varepsilon_{\delta_k}})(\partial f_r(v_{\varepsilon_\delta,\tau}))_k\,d\sigma_x\,d\theta\right] \\ &+\sum_{l=1}^{n}\int_0^t\int_{\Omega_\varepsilon^p}\left\langle D\frac{\partial}{\partial x_l}v_{\varepsilon_\delta}, \frac{\partial}{\partial x_l}(\partial f_r(v_{\varepsilon_\delta,\tau}))\right\rangle_{I_2}dx\,d\theta + \int_0^t\int_{\Omega_\varepsilon^p}\langle\vec{q}_\varepsilon\cdot\nabla v_{\varepsilon_\delta}, \partial f_r(v_{\varepsilon_\delta,\tau})\rangle_{I_2}\,dx\,d\theta \\ &= \int_0^t\left\langle S_2\bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}), \partial f_r(v_{\varepsilon_\delta,\tau})\right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}\times[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}}\,d\theta + \lambda\kappa\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} v_{\varepsilon_{\delta_k}}\partial f_r(v_{\varepsilon_\delta,\tau})_k\,dx\,d\theta,\end{aligned}$$

i.e.,

$$
\begin{aligned}
&\int_0^t \langle \partial_\theta v_{\varepsilon\delta}, \partial f_r(v_{\varepsilon\delta,\tau})\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}\times[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}}\,d\theta \\
&= -\sum_{l=1}^n \int_0^t \int_{\Omega_\varepsilon^p} \left\langle D\frac{\partial}{\partial x_l} v_{\varepsilon\delta}, \frac{\partial}{\partial x_l}(\partial f_r(v_{\varepsilon\delta,\tau}))\right\rangle_{I_2} dx\,d\theta \\
&+\sum_{k=1}^{I_2}\left[\int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n} v_{\varepsilon\delta_k}\,(\partial f_r(v_{\varepsilon\delta,\tau}))_k\,ds\,d\theta + \lambda\varepsilon k_d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon\delta_k})\,(\partial f_r(v_{\varepsilon\delta,\tau}))_k\,d\sigma_x\,d\theta\right] \\
&-\int_0^t \int_{\Omega_\varepsilon^p} \langle \vec{q}_\varepsilon\cdot\nabla v_{\varepsilon\delta}, \partial f_r(v_{\varepsilon\delta,\tau})\rangle_{I_2}\,dx\,d\theta + \int_0^t \left\langle S_2\bar{R}(\hat{u}_{\varepsilon\delta}, v_{\varepsilon\delta}), \partial f_r(v_{\varepsilon\delta,\tau})\right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}\times[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}} d\theta \\
&-(1-\lambda)\kappa\sum_{k=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} v_{\varepsilon\delta_k}\,\partial f_r(v_{\varepsilon\delta,\tau})_k\,dx\,d\theta \\
&=: I_{diff}^{(t)} + I_{bound}^{(t)} + I_{advec}^{(t)} + I_{reac}^{(t)} + I_{Ex}^{(t)} \qquad \text{for a.e. } t,
\end{aligned}
\tag{4.2.81}
$$

where

$$
\begin{aligned}
I_{diff}^{(t)} &:= -\sum_{l=1}^n \int_0^t \int_{\Omega_\varepsilon^p} \left\langle D\frac{\partial}{\partial x_l} v_{\varepsilon\delta}, \frac{\partial}{\partial x_l}(\partial f_r(v_{\varepsilon\delta,\tau}))\right\rangle_{I_2} dx\,d\theta, \\
I_{bound}^{(t)} &:= \sum_{k=1}^{I_2}\left[\int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n} v_{\varepsilon\delta_k}\,(\partial f_r(v_{\varepsilon\delta,\tau}))_k\,ds\,d\theta + \lambda\varepsilon k_d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon\delta_k})\,(\partial f_r(v_{\varepsilon\delta,\tau}))_k\,d\sigma_x\,d\theta\right], \\
I_{advec}^{(t)} &:= -\int_0^t \int_{\Omega_\varepsilon^p} \langle \vec{q}_\varepsilon\cdot\nabla v_{\varepsilon\delta}, \partial f_r(v_{\varepsilon\delta,\tau})\rangle_{I_2}\,dx\,d\theta, \\
I_{reac}^{(t)} &:= \int_0^t \left\langle S_2\bar{R}(\hat{u}_{\varepsilon\delta}, v_{\varepsilon\delta}), \partial f_r(v_{\varepsilon\delta,\tau})\right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_2}\times[H^{1,q}(\Omega_\varepsilon^p)]^{I_2}} d\theta, \\
I_{Ex}^{(t)} &:= -(1-\lambda)\kappa\sum_{k=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} v_{\varepsilon\delta_k}\,\partial f_r(v_{\varepsilon\delta,\tau})_k\,dx\,d\theta.
\end{aligned}
$$

Now we simplify and estimate the terms $I_{diff}^{(t)}$, $I_{bound}^{(t)}$, $I_{advec}^{(t)}$, $I_{reac}^{(t)}$ and $I_{Ex}^{(t)}$ one by one. With the help of (4.2.9), the term $I_{reac}^{(t)}$ can be estimated in the same way as we did in the lemma 4.1.1.2.7 and this will give

$$
I_{reac}^{(t)} \le \lambda\, r\, C \sum_{k=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} \tau\left[\left|\mu_k^0\right| + T\,|\Omega|\,|\log\tau| + v_{\varepsilon\delta_k,\tau}\right] dx\,d\theta =: h(t,\tau,v_{\varepsilon\delta,\tau}) \qquad \text{for a.e. } t,
$$

where C is independent of λ, $\hat{u}_{\varepsilon\delta}$ and $v_{\varepsilon\delta,\tau}$ and all the other terms of $h(t,\tau,v_{\varepsilon\delta,\tau})$ are bounded and tend to zero as $\tau\to 0$ for a.e. t, i.e.,

$$
I_{reac}^{(t)} \le h(t,\tau,v_{\varepsilon\delta,\tau}) \to 0 \text{ as } \tau\to 0 \quad \text{for a.e. } t. \tag{4.2.82}
$$

$$
\begin{aligned}
I_{Ex}^{(t)} &= -\kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} v_{\varepsilon\delta_k}\,\partial f_r(v_{\varepsilon\delta},\tau)_k\,dx\,d\theta \\
&= \kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} r(\tau - v_{\varepsilon\delta_k,\tau}) f_{r-1}(v_{\varepsilon\delta},\tau)(\mu_k^0 + \log v_{\varepsilon\delta_k,\tau})\,dx\,d\theta \quad \text{since } v_{\varepsilon\delta_k,\tau} = v_{\varepsilon\delta_k} + \tau \\
&= \tau\kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} r(\mu_k^0 + \log v_{\varepsilon\delta_k,\tau}) f_{r-1}(v_{\varepsilon\delta,\tau})\,dx\,d\theta \\
&\qquad + r\kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t \int_{\Omega_\varepsilon^p} -v_{\varepsilon\delta_k,\tau}(\mu_k^0 + \log v_{\varepsilon\delta_k,\tau}) f_{r-1}(v_{\varepsilon\delta,\tau})\,dx\,d\theta.
\end{aligned}
\tag{4.2.83}
$$

It can be shown that

$$-v_{\varepsilon_{\delta_k},\tau}(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}) \le e^{-(1+\mu_k^0)} \qquad \forall i. \tag{4.2.84}$$

We have $\log v_{\varepsilon_{\delta_k},\tau} \le v_{\varepsilon_{\delta_k},\tau} \le g_k(v_{\varepsilon_{\delta_k},\tau})$ and $g_k(v_{\varepsilon_{\delta_k},\tau}) \ge (e-1)e^{-\mu_k^0}$. Choosing a constant $C = \max_{1\le k\le I_2}\left(1+|\mu_k^0|\, e^{-\mu_k^0}(e-1)\right)$, we obtain

$$\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau} \le \mu_k^0+g_k(v_{\varepsilon_{\delta_k},\tau}) \le \left|\mu_k^0\right|+g_k(v_{\varepsilon_{\delta_k},\tau}) \le C\, g_k(v_{\varepsilon_{\delta_k},\tau}). \tag{4.2.85}$$

Combining (4.2.83), (4.2.84) and (4.2.85), we get

$$\begin{aligned}
I_{Ex}^{(t)} &\le (1-\lambda)\left[r\tau\kappa\sum_{i=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} C g_k(v_{\varepsilon_{\delta_k},\tau})f_{r-1}(v_{\varepsilon_\delta,\tau})\,dx\,d\theta+\kappa\sum_{i=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} re^{-(1+\mu_k^0)}f_{r-1}(v_{\varepsilon_\delta,\tau})\,dx\,d\theta\right]\\
&\le r\tau\kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} C g_k(v_{\varepsilon_{\delta_k},\tau})f_{r-1}(v_{\varepsilon_\delta,\tau})\,dx\,d\theta\\
&\quad+\kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} r(e(e-1))^{-1}g_k(v_{\varepsilon_{\delta_k},\tau})f_{r-1}(v_{\varepsilon_\delta,\tau})\,dx\,d\theta\\
&\le r\tau\kappa(1-\lambda)C\sum_{i=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} g(v_{\varepsilon_\delta,\tau})f_{r-1}(v_{\varepsilon_\delta,\tau})\,dx\,d\theta\\
&\quad+\kappa(1-\lambda)\sum_{i=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} r(e(e-1))^{-1}g(v_{\varepsilon_\delta,\tau})f_{r-1}(v_{\varepsilon_\delta,\tau})\,dx\,d\theta \quad \text{since } g_k(v_{\varepsilon_{\delta_k},\tau})\le g(v_{\varepsilon_\delta,\tau})\\
&\le I_2 r\kappa\tau\, C\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta+I_2r\kappa(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta \quad \text{since } 0\le\lambda\le 1\\
&\qquad \text{and } f_r=f_{r-1}g \quad \text{for a.e. } t.
\end{aligned}$$

As $\tau\to 0$, $f_r(v_{\varepsilon_\delta,\tau})$ is bounded in $L^1((0,T)\times\Omega_\varepsilon^p)$. Therefore for a.e. t the first term in the r.h.s. of the above inequality tends to zero as $\tau\to 0$. Denote the first term by $l(t,\tau,v_{\varepsilon_\delta,\tau})$, then

$$I_{Ex}^{(t)} \le l(t,\tau,v_{\varepsilon_\delta,\tau})+I_2rk(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta \qquad \text{for a.e. } t. \tag{4.2.86}$$

Again,

$$\begin{aligned}
I_{advec}^{(t)} &= -\int_0^t\int_{\Omega_\varepsilon^p}\langle\vec{q}_\varepsilon\cdot\nabla v_{\varepsilon_\delta},\partial f_r(v_{\varepsilon_\delta,\tau})\rangle_{I_2}\,dx\,d\theta\\
&= -\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\vec{q}_\varepsilon\cdot\nabla v_{\varepsilon_{\delta_k}}\left(\partial f_r(v_{\varepsilon_\delta,\tau})\right)_k\,dx\,d\theta\\
&= -\sum_{k=1}^{I_2}\sum_{l=1}^{n}\int_0^t\int_{\Omega_\varepsilon^p} q_l\frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\left(\partial f_r(v_{\varepsilon_\delta,\tau})\right)_k\,dx\,d\theta\\
&= -\sum_{l=1}^{n}\int_0^t\int_{\Omega_\varepsilon^p}\frac{\partial f_r(v_{\varepsilon_\delta,\tau})}{\partial x_l}q_l\,dx\,d\theta\\
&= -\int_0^t\int_{\Omega_\varepsilon^p}\nabla_x f_r(v_{\varepsilon_\delta,\tau})\cdot\vec{q}_\varepsilon\,dx\,d\theta\\
&= \int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\nabla\cdot\vec{q}_\varepsilon\,dx\,d\theta-\int_0^T\int_{\partial\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\vec{q}_\varepsilon\cdot\vec{n}\,ds\,d\theta\\
&= -\int_0^t\int_{\partial\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\vec{q}_\varepsilon\cdot\vec{n}\,ds\,d\theta,\ \text{since } \nabla\cdot\vec{q}_\varepsilon=0 \text{ in } \Omega_\varepsilon^p
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t \int_{\partial\Omega} f_r(v_{\varepsilon\delta,\tau})\vec{q}_\varepsilon \cdot \vec{n}\, ds\, d\theta - \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon\delta,\tau})\vec{q}_\varepsilon \cdot \vec{n}\, d\sigma_x\, d\theta \\
&= -\int_0^t \int_{\partial\Omega} f_r(v_{\varepsilon\delta,\tau})\vec{q}_\varepsilon \cdot \vec{n}\, ds\, d\theta, \text{ since } \vec{q}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \\
&= -\int_0^t \int_{\partial\Omega_{in}} f_r(v_{\varepsilon\delta,\tau})\vec{q}_\varepsilon \cdot \vec{n}\, ds\, d\theta - \int_0^t \int_{\partial\Omega_{out}} f_r(v_{\varepsilon\delta,\tau})\vec{q}_\varepsilon \cdot \vec{n}\, ds\, d\theta \\
&\le -\int_0^t \int_{\partial\Omega_{in}} f_r(v_{\varepsilon\delta,\tau})\vec{q}_\varepsilon \cdot \vec{n}\, ds\, d\theta, \text{ since } \vec{q}_\varepsilon \cdot \vec{n} \ge 0 \text{ on } \partial\Omega_{out} \text{ and } f_r(v_{\varepsilon\delta,\tau}) \ge 0 \\
&\le \|\vec{q}_\varepsilon \cdot \vec{n}\|_{L^\infty((0,T)\times\partial\Omega_{in})} \int_0^t \int_{\partial\Omega_{in}} |f_r(v_{\varepsilon\delta,\tau})|\, ds\, d\theta \\
&= C_{26} \int_0^t \int_{\partial\Omega} f_r(v_{\varepsilon\delta,\tau})\, ds\, d\theta \qquad \text{for a.e. } t,
\end{aligned} \tag{4.2.87}
$$

where $C_{26} := \|\vec{q}_\varepsilon \cdot \vec{n}\|_{L^\infty((0,T)\times\partial\Omega_{in})}$ and $f_r \ge 0$. Note that C_{26} is independent of ε, δ, λ, τ and t. Again,

$$
\begin{aligned}
I^{(t)}_{bound} &= \sum_{k=1}^{I_2} \left[\int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon\delta_k} \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta + \lambda\varepsilon k_d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon\delta_k}) \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k d\sigma_x\, d\theta \right] \\
&= \sum_{k=1}^{I_2} \left[\int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} \left(v_{\varepsilon\delta_k,\tau} - \tau \right) \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta \right. \\
&\qquad \left. + \lambda\varepsilon k_d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon\delta_k}) \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k d\sigma_x\, d\theta \right] \\
&= \sum_{k=1}^{I_2} \left[\int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon\delta_k,\tau} \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta - \int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} \tau \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta \right] \\
&\quad + \sum_{k=1}^{I_2} \left[\lambda\varepsilon k_d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon\delta_k}) \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k d\sigma_x\, d\theta \right] \\
&=: \sum_{k=1}^{I_2} \left[\text{Boundary}_{1,k} + \text{Boundary}_{2,k} + \text{Boundary}_{3,k} \right],
\end{aligned} \tag{4.2.88}
$$

where

$$
\text{Boundary}_{1,k} := \int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon\delta_k,\tau} \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta, \tag{4.2.89}
$$

$$
\text{Boundary}_{2,k} := -\int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} \tau \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta \text{ and} \tag{4.2.90}
$$

$$
\text{Boundary}_{3,k} := \lambda\varepsilon k_d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon\delta_k}) \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k d\sigma_x\, d\theta. \tag{4.2.91}
$$

Now,

$$
\begin{aligned}
\text{Boundary}_{1,k} &= \int_0^t \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon\delta_k,\tau} \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta \\
&= \int_0^t \int_{\partial\Omega_{in}} -|\vec{q}_\varepsilon \cdot \vec{n}|\, v_{\varepsilon\delta_k,\tau} \left(\partial f_r(v_{\varepsilon\delta,\tau})\right)_k ds\, d\theta \\
&= \int_0^t \int_{\partial\Omega_{in}} -r f_{r-1}(v_{\varepsilon\delta,\tau})\, |\vec{q}_\varepsilon \cdot \vec{n}|\, v_{\varepsilon\delta_k,\tau} \left(\mu_k^0 + \log v_{\varepsilon\delta_k,\tau} \right) ds\, d\theta.
\end{aligned}
$$

It can be shown that $-v_{\varepsilon_{\delta_k},\tau}\left(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\right)\leq \frac{1}{e(e-1)}g_k(v_{\varepsilon_{\delta_k},\tau})$. This gives[30]

$$
\begin{aligned}
\text{Boundary}_{1,k} &\leq \int_0^t\int_{\partial\Omega_{in}} r f_{r-1}\frac{1}{e(e-1)}g_k(v_{\varepsilon_{\delta_k},\tau})\,|\vec{q}_\varepsilon\cdot\vec{n}|\,ds\,d\theta \\
&\leq r\,||\vec{q}_\varepsilon\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}\frac{1}{e(e-1)}\int_0^t\int_{\partial\Omega_{in}} f_{r-1}(v_{\varepsilon_\delta,\tau})g_k(v_{\varepsilon_\delta,\tau})\,ds\,d\theta \\
&\leq r\,||\vec{q}_\varepsilon\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}\frac{1}{e(e-1)}\int_0^t\int_{\partial\Omega_{in}} f_{r-1}(v_{\varepsilon_\delta,\tau})g(v_{\varepsilon_\delta,\tau})\,ds\,d\theta,\ \text{since } g_k\leq g \\
&= C_{27}\int_0^t\int_{\partial\Omega_{in}} f_r(v_{\varepsilon_\delta,\tau}(t,x))\,ds\,d\theta, \qquad (4.2.92)
\end{aligned}
$$

where C_{27} $(:= r\,||\vec{q}_\varepsilon\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}\frac{1}{e(e-1)})$ is independent of ε, δ, λ, τ and t.

$$
\begin{aligned}
\text{Boundary}_{2,k} &:= -\int_0^t\int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n}\,\tau\left(\partial f_r(v_{\varepsilon_\delta,\tau})\right)_k\,ds\,d\theta \\
&= \int_0^t\int_{\partial\Omega_{in}} |\vec{q}_\varepsilon\cdot\vec{n}|\,\tau r f_{r-1}(v_{\varepsilon_\delta,\tau})\left(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\right)ds\,d\theta.
\end{aligned}
$$

Let $\partial\Omega_{in}^+ := \left\{x\in\partial\Omega_{in} : \mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\geq 0\right\}$ and $\partial\Omega_{in}^- := \left\{x\in\partial\Omega_{in} : \mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\leq 0\right\}$. On the boundary $\partial\Omega_{in}^-$, the integrand is nonpositive and it can be estimated by zero. This gives

$$
\text{Boundary}_{2,k} \leq \int_0^t\int_{\partial\Omega_{in}^+} |\vec{q}_\varepsilon\cdot\vec{n}|\,\tau r f_{r-1}(v_{\varepsilon_\delta,\tau})\left(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\right)ds\,d\theta.
$$

From the definition of g_k, $(e-1)e^{-\mu_k^0}\leq g_k(v_{\varepsilon_{\delta_k},\tau})$ and it can be shown that $\log v_{\varepsilon_{\delta_k},\tau}\leq v_{\varepsilon_{\delta_k},\tau}\leq g_k(v_{\varepsilon_{\delta_k},\tau})$. Choosing $C_{28} := \max_{1\leq k\leq I_2}\left(1+\left|\mu_0^k\right|e^{\mu_k^0}(e-1)^{-1}\right)$, we have

$$
\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\leq C_{28}g_k(v_{\varepsilon_{\delta_k},\tau})\leq C_{28}g(v_{\varepsilon_\delta,\tau}).
$$

$$
\begin{aligned}
\text{Boundary}_{2,k} &\leq \int_0^t\int_{\partial\Omega_{in}^+} |\vec{q}_\varepsilon\cdot\vec{n}|\,\tau r f_{r-1}(v_{\varepsilon_\delta,\tau})C_{28}g(v_{\varepsilon_\delta,\tau})\,ds\,d\theta \\
&\leq C_{28}r\tau\,||\vec{q}_\varepsilon\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}\int_0^t\int_{\partial\Omega_{in}} f_r(v_{\varepsilon_\delta,\tau})\,ds\,d\theta \\
&= C_{29}\int_0^T\int_{\partial\Omega_{in}} f_r(v_{\varepsilon_\delta,\tau})\,ds\,d\theta, \qquad (4.2.93)
\end{aligned}
$$

where C_{29} $(:= C_{28}r\tau\,||\vec{q}_\varepsilon\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})})$ is independent of ε, δ, λ and t.

$$
\begin{aligned}
\text{Boundary}_{3,k} &= \lambda\varepsilon k_d\int_0^t\int_{\Gamma_\varepsilon}\psi_\delta(w_{\varepsilon_{\delta_k}})\left(\partial f_r(v_{\varepsilon_\delta,\tau})\right)_k\,d\sigma_x\,d\theta \\
&\leq k_d\varepsilon\int_0^t\int_{\Gamma_\varepsilon}\left(\partial f_r(v_{\varepsilon_\delta,\tau})\right)_k\,d\sigma_x\,d\theta,\text{[31]} \\
&= k_d\varepsilon\int_0^t\int_{\Gamma_\varepsilon} r f_{r-1}(v_{\varepsilon_{\delta_k},\tau})\left(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\right)d\sigma_x\,d\theta
\end{aligned}
$$

[30] The simplification of the terms $\text{Boundary}_{1,k}$ and $\text{Boundary}_{2,k}$ are imitated from [Krä08].
[31] Note that $0\leq\lambda\leq 1$, $\psi_\delta(v_{\varepsilon_{\delta_k}})\leq 1$ and $\varepsilon\ll 1$.

$$\leq \varepsilon k_d C_{28} r \int_0^t \int_{\Gamma_\varepsilon} f_{r-1}(v_{\varepsilon_\delta,\tau}) g(v_{\varepsilon_\delta,\tau})\, d\sigma_x\, d\theta,^{32}$$
$$= C_{30}\varepsilon \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\, d\sigma_x\, d\theta, \tag{4.2.94}$$

where C_{30} $(:= k_d C_{28} r)$ is independent of ε, δ, λ and t. Substituting the estimates for $\text{Boundary}_{p,k}$ for $1 \leq p \leq 3$ in (4.2.88), we obtain[33]

$$\begin{aligned}
I^{(t)}_{bound} &= \sum_{k=1}^{I_2} \Big[\text{Boundary}_{1,k} + \text{Boundary}_{2,k} + \text{Boundary}_{3,k}\Big] \\
&\leq \sum_{k=1}^{I_2} C_{31} \left[\int_0^t \int_{\partial\Omega_{in}} f_r(v_{\varepsilon_\delta,\tau})\, ds\, d\theta + \int_0^t \int_{\partial\Omega_{in}} f_r(v_{\varepsilon_\delta,\tau})\, ds\, d\theta + \varepsilon \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\, d\sigma_x\, d\theta\right] \\
&= 2C_{31} I_2 \int_0^t \int_{\partial\Omega_{in}} f_r(v_{\varepsilon_\delta,\tau})\, ds\, d\theta + \varepsilon C_{31} I_2 \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\, d\sigma_x\, d\theta \\
&\leq C_{32} \left[\int_0^t \int_{\partial\Omega} f_r(v_{\varepsilon_\delta,\tau})\, ds\, d\theta + \varepsilon \int_0^T \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\, d\sigma_x\, d\theta\right] \qquad \text{for a.e. } t.
\end{aligned} \tag{4.2.95}$$

The term I_{diff} can be estimated as in lemma 4.1.1.2.7.

$$\begin{aligned}
I^{(t)}_{diff} = &-D \sum_{k=1}^{I_2} \sum_{l=1}^{n} \int_0^t \int_{\Omega^p_\varepsilon} r(r-1) f_{r-2} \sum_{\upsilon=1}^{I_2} \left(\mu^0_\upsilon + \log v_{\varepsilon_{\delta_\upsilon},\tau}\right)\left(\mu^0_k + \log v_{\varepsilon_{\delta_k},\tau}\right) \frac{\partial v_{\varepsilon_{\delta_\upsilon},\tau}}{\partial x_l} \frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\, dx\, d\theta \\
&\underbrace{- D \sum_{k=1}^{I_2} \sum_{l=1}^{n} \int_0^t \int_{\Omega^p_\varepsilon} r f_{r-1} \frac{1}{v_{\varepsilon_{\delta_k}}} \frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l} \frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\, dx\, d\theta}_{\leq 0} \\
\leq &-D \sum_{l=1}^{n} \int_0^t \int_{\Omega^p_\varepsilon} r(r-1) f_{r-2} \left(\sum_{k=1}^{I_2} \left(\mu^0_k + \log v_{\varepsilon_{\delta_k},\tau}\right) \frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\right)^2 dx\, d\theta \qquad \text{for a.e. } t.
\end{aligned} \tag{4.2.96}$$

Combining (4.2.81), (4.2.82), (4.2.86), (4.2.87), (4.2.95) and (4.2.96), we get[34]

$$\begin{aligned}
&\int_0^t \langle \partial_\theta v_{\varepsilon_\delta}, \partial f_r(v_{\varepsilon_\delta,\tau})\rangle_{[H^{1,q}(\Omega^p_\varepsilon)]^{*I_2} \times [H^{1,q}(\Omega^p_\varepsilon)]^{I_2}}\, d\theta \\
&= I^{(t)}_{diif} + I^{(t)}_{bound} + I^{(t)}_{advec} + I^{(t)}_{reac} + I^{(t)}_{Ex} \\
&\leq -D \sum_{l=1}^{n} \int_0^t \int_{\Omega^p_\varepsilon} r(r-1) f_{r-2} \left(\sum_{k=1}^{I_2} \left(\mu^0_k + \log v_{\varepsilon_{\delta_k},\tau}\right) \frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\right)^2 dx\, d\theta \\
&\quad + C_{32} \left[\int_0^t \int_{\partial\Omega} f_r(v_{\varepsilon_\delta,\tau})\, ds\, d\theta + \varepsilon \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\, d\sigma_x\, d\theta\right] + C_{26} \int_0^t \int_{\partial\Omega} f_r(v_{\varepsilon_\delta,\tau})\, ds\, d\theta \\
&\qquad + h(t,\tau,v_{\varepsilon_\delta,\tau}) + l(t,\tau,v_{\varepsilon_\delta,\tau}) + I_2 r k (e(e-1))^{-1} \int_0^t \int_{\Omega^p_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\, dx\, d\theta
\end{aligned}$$

[32] See the above calculation for $\text{Boundary}_{2,k}$.
[33] Where $C_{31} = \max(C_{27}, C_{29}, C_{30})$ and $C_{32} = I_2 \max(2C_{31}, C_{31})$.
[34] The idea to further estimate the term $I_{diff} + I_{bound} + I_{advec} + I_{reac} + I_{Ex}$ is borrowed from [Krä08]. We also note that $f_r(u) = [g(u)]^r = [g^{\frac{r}{2}}(u)]^2 = f^2_{\frac{r}{2}}(u)$.

$$
\begin{aligned}
&= -D\sum_{l=1}^{n}\int_0^t\int_{\Omega_\varepsilon^p} r(r-1)f_{r-2}\left(\sum_{k=1}^{I_2}\left(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\right)\frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\right)^2 dx\,d\theta \\
&\quad + (C_{32}+C_{26})\int_0^t\int_{\partial\Omega} f_r(v_{\varepsilon_\delta,\tau})\,ds\,d\theta + C_{32}\varepsilon\int_0^t\int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_x\,d\theta \\
&\qquad + h(t,\tau,v_{\varepsilon_\delta,\tau}) + l(t,\tau,v_{\varepsilon_\delta,\tau}) + I_2 rk(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta \\
&= -D\sum_{l=1}^{n}\int_0^t\int_{\Omega_\varepsilon^p} r(r-1)f_{r-2}\left(\sum_{k=1}^{I_2}\left(\mu_k^0+\log v_{\varepsilon_{\delta_k},\tau}\right)\frac{\partial v_{\varepsilon_{\delta_k}}}{\partial x_l}\right)^2 dx\,d\theta \\
&\quad + (C_{32}+C_{26})\int_0^t\int_{\partial\Omega} f_{\frac{r}{2}}^2(v_{\varepsilon_\delta,\tau})\,ds\,d\theta + C_{32}\varepsilon\int_0^t\int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_x\,d\theta \\
&\qquad + h(t,\tau,v_{\varepsilon_\delta,\tau}) + l(t,\tau,v_{\varepsilon_\delta,\tau}) + I_2 rk(e(e-1))^{-1}\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta
\end{aligned} \tag{4.2.97}
$$

for a.e. t, where C_{26} and C_{32} are independent of ε, δ, λ and t. We further simplify the terms in (4.2.97).

$$
\begin{aligned}
& C_{32}\varepsilon\int_0^t\int_{\Gamma_\varepsilon} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_x\,d\theta \\
= \;& C_{32}\varepsilon\int_0^t\int_{\cup_{k\in\mathbb{Z}^n}\varepsilon\Gamma_k} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_x\,d\theta \\
= \;& C_{32}\varepsilon\int_0^t\int_{\cup_{k\in\mathbb{Z}^n}\Gamma_k} f_r(v_{\varepsilon_\delta,\tau})\varepsilon^{n-1}\,d\sigma_y\,d\theta \\
= \;& C_{32}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\int_0^t\int_{\Gamma_k} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_y\,d\theta \\
= \;& C_{32}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\int_0^t\int_{\Gamma_k} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_y\,d\theta\times\frac{1}{|Y_k^p|}\int_{Y_k^p} dy \\
= \;& C_{32}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\frac{1}{|Y_k^p|}\int_0^t\int_{Y_k^p}\int_{\Gamma_k} f_r(v_{\varepsilon_\delta,\tau})\,d\sigma_y\,d\theta\,dy \\
= \;& C_{32}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\frac{1}{|Y_k^p|}\int_0^t\int_{Y_k^p} f_r(v_{\varepsilon_\delta,\tau})\,d\theta\,dy\times\int_{\Gamma_k} d\sigma_y \\
= \;& C_{32}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\frac{1}{|Y_k^p|}\int_0^t\int_{Y_k^p} f_r(v_{\varepsilon_\delta,\tau})\,d\theta\,dy\times|\Gamma_k| \\
= \;& C_{32}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\frac{|\Gamma_k|}{|Y_k^p|}\int_0^t\int_{Y_k^p} f_r(v_{\varepsilon_\delta,\tau})\,d\theta\,dy \\
= \;& C_{32}\frac{|\Gamma|}{|Y^p|}\varepsilon^n\sum_{k\in\mathbb{Z}^n}\int_0^t\int_{Y_k^p} f_r(v_{\varepsilon_\delta,\tau})\,d\theta\,dy \qquad \text{since } |Y_k^p|=|Y^p| \text{ and } |\Gamma_k|=|\Gamma| \\
= \;& C_{32}\frac{|\Gamma|}{|Y^p|}\varepsilon^n\int_0^t\int_{\cup_{k\in\mathbb{Z}^n}Y_k^p} f_r(v_{\varepsilon_\delta,\tau})\,d\theta\,dy \\
= \;& C_{32}\frac{|\Gamma|}{|Y^p|}\int_0^t\int_{\cup_{k\in\mathbb{Z}^n}\varepsilon Y_k^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta \\
= \;& C_{32}\frac{|\Gamma|}{|Y^p|}\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta.
\end{aligned} \tag{4.2.98}
$$

Also using theorem 3.4.3.2 we have

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} f^2_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t,x))\,ds\,d\theta \\
=&\int_0^t \left|\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|\right|^2_{L^2(\partial\Omega)}\,d\theta \\
\le& C_8\left(\int_0^t \left|\left|\nabla f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|\right|_{[L^2(\Omega^p_\varepsilon)]^n}\left|\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|\right|_{L^2(\Omega^p_\varepsilon)} + \left|\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t)\right|\right|^2_{L^2(\Omega^p_\varepsilon)}\right) \\
\le& C_8\int_0^t\left(\underbrace{\varsigma\left|\left|\nabla f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau})\right|\right|^2_{[L^2(\Omega^p_\varepsilon)]^n} + \hat{\Lambda}_\varsigma\left|\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|\right|^2_{L^2(\Omega^p_\varepsilon)}}_{\text{due to Young's inequality}} + \left|\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|\right|^2_{L^2(\Omega^p_\varepsilon)}\right)d\theta \\
=& C_8\int_0^t\left(\varsigma\left|\left|\nabla f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau})\right|\right|^2_{[L^2(\Omega^p_\varepsilon)]^n} + \Lambda_\varsigma\left|\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|\right|^2_{L^2(\Omega^p_\varepsilon)}\right)d\theta,\ \Lambda_\varsigma = \hat{\Lambda}_\varsigma + 1,
\end{aligned} \tag{4.2.99}$$

where C_8 (independent of ε and ς) is a constant in Young's inequality which will be chosen later. Further note that

$$\begin{aligned}
\left|\left|\nabla f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau})\right|\right|^2_{[L^2(\Omega^p_\varepsilon)]^n} &= \sum_{l=1}^n \left|\left|\frac{\partial f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau})}{\partial x_l}\right|\right|^2_{L^2(\Omega^p_\varepsilon)} \\
&= \frac{r^2}{4}\int_{\Omega^p_\varepsilon} f_{r-2}(v_{\varepsilon\delta,\tau})\sum_{l=1}^n\left(\sum_{k=1}^{I_2}\left(\mu^0_k + \log v_{\varepsilon\delta_k}\right)\frac{\partial v_{\varepsilon\delta_k}}{\partial x_l}\right)^2 dx.
\end{aligned} \tag{4.2.100}$$

Combining (4.2.97), (4.2.98), (4.2.99) and (4.2.100), we obtain

$$\begin{aligned}
&\int_0^t \langle\partial_\theta v_{\varepsilon\delta}, \partial f_r(v_{\varepsilon\delta,\tau})\rangle_{[H^{1,q}(\Omega^p_\varepsilon)]^{*I_2}\times[H^{1,q}(\Omega^p_\varepsilon)]^{I_2}}\,d\theta \\
&\le -D\sum_{l=1}^n\int_0^t\int_{\Omega^p_\varepsilon} r(r-1)f_{r-2}\left(\sum_{k=1}^{I_2}\left(\mu^0_k + \log v_{\varepsilon\delta_k,\tau}\right)\frac{\partial v_{\varepsilon\delta_k}}{\partial x_l}\right)^2 dx\,d\theta \\
&\quad + (C_{32}+C_{26})C_8\int_0^t\int_{\Omega^p_\varepsilon}\frac{r^2}{4}f_{r-2}\varsigma\sum_{l=1}^n\left(\sum_{k=1}^{I_2}\left(\mu^0_k + \log v_{\varepsilon\delta_k}\right)\frac{\partial v_{\varepsilon\delta_k}}{\partial x_l}\right)^2 dx\,d\theta \\
&\quad + (C_{32}+C_{26})C_8\int_0^t\int_{\Omega^p_\varepsilon}\Lambda_\varsigma\left|f_{\frac{r}{2}}(v_{\varepsilon\delta,\tau}(t))\right|^2_{L^2(\Omega^p_\varepsilon)}dx\,d\theta + C_{32}\frac{|\Gamma|}{|Y^p|}\int_0^t\int_{\Omega^p_\varepsilon} f_r(v_{\varepsilon\delta,\tau})\,dx\,d\theta \\
&\quad + h(t,\tau,v_{\varepsilon\delta,\tau}) + l(t,\tau,v_{\varepsilon\delta,\tau}) + I_2rk(e(e-1))^{-1}\int_0^t\int_{\Omega^p_\varepsilon} f_r(v_{\varepsilon\delta,\tau})\,dx\,d\theta \\
&\le \left[-Dr(r-1)+\varsigma C_{33}\frac{r^2}{4}\right]\int_0^t\int_{\Omega^p_\varepsilon} f_{r-2}\sum_{l=1}^n\left(\sum_{k=1}^{I_2}\left(\mu^0_k + \log v_{\varepsilon\delta_k}\right)\frac{\partial v_{\varepsilon\delta_k}}{\partial x_l}\right)^2 dx\,d\theta + h(t,\tau,v_{\varepsilon\delta,\tau}) \\
&\quad + l(t,\tau,v_{\varepsilon\delta,\tau}) + \left(C_{33}\Lambda_\varsigma + C_{32}\frac{|\Gamma|}{|Y^p|} + I_2r\kappa(e(e-1))^{-1}\right)\int_0^t\int_{\Omega^p_\varepsilon} f_r(v_{\varepsilon\delta,\tau})\,dx\,d\theta \quad \text{for a.e. } t.^{35}
\end{aligned} \tag{4.2.101}$$

Choosing $\varsigma \le \frac{4D(r-1)}{C_{33}r}$, this shows that $\Lambda_\varsigma + 1$ is independent of ε, λ and δ. This gives

$$\begin{aligned}
&\int_0^t \langle\partial_\theta v_{\varepsilon\delta}, \partial f_r(v_{\varepsilon\delta,\tau})\rangle_{[H^{1,q}(\Omega^p_\varepsilon)]^{*I_2}\times[H^{1,q}(\Omega^p_\varepsilon)]^{I_2}}\,d\theta \\
&\le h(t,\tau,v_{\varepsilon\delta,\tau}) + l(t,\tau,v_{\varepsilon\delta,\tau}) + \left(C_{33}\Lambda_\varsigma + C_{32}\frac{|\Gamma|}{|Y^p|} + I_2r\kappa(e(e-1))^{-1}\right)\int_0^t\int_{\Omega^p_\varepsilon} f_r(v_{\varepsilon\delta,\tau})\,dx\,d\theta
\end{aligned}$$

[35] Where $C_{33} = C_8\,(C_{26}+C_{32})$.

$$\leq h(t,\tau,v_{\varepsilon_\delta,\tau})+l(t,\tau,v_{\varepsilon_\delta,\tau})+C_{34}\int_0^t\int_{\Omega_\varepsilon^p} f_r(v_{\varepsilon_\delta,\tau})\,dx\,d\theta$$
$$\leq h(t,\tau,v_{\varepsilon_\delta,\tau})+l(t,\tau,v_{\varepsilon_\delta,\tau})+C_{34}\int_0^t F_r(v_{\varepsilon_\delta,\tau})\,d\theta \quad \text{for a.e. } t. \tag{4.2.102}$$

where $C_{34}\left(:=\left(C_{33}\Lambda_\varsigma+C_{32}\frac{|\Gamma|}{|Y^p|}+I_2 r\kappa(e(e-1))^{-1}\right)\right)$, and $h(t,\tau,v_{\varepsilon_\delta,\tau})$ and $l(t,\tau,v_{\varepsilon_\delta,\tau})$ tend to zero as $\tau\to 0$ for a.e. t. ♦

Proof of theorem 4.2.1.3.5. Let v_{ε_δ} be a solution of the problem $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$. Since we only the know the nonnegativity of v_{ε_δ}, let $v_{\varepsilon_\delta,\tau}:=v_{\varepsilon_\delta}+\tau$ for $\tau>0$. Clearly $v_{\varepsilon_\delta,\tau}\in\mathcal{G}_\varepsilon^v$. Here also we introduce the regularization of $v_{\varepsilon_\delta,\tau}$ and replicating the steps of theorem 4.1.1.2.3, we obtain an inequality similar to (4.1.34), i.e.,

$$|F_r(v_{\varepsilon_\delta,\tau}(t))-F_r(v_{\varepsilon_\delta,\tau}(0))|\leq h(t,\tau,v_{\varepsilon_\delta,\tau})+l(t,\tau,v_{\varepsilon_\delta,\tau})+C_{34}\int_0^t F_r(v_{\varepsilon_\delta,\tau})\,d\theta$$
$$\Longrightarrow F_r(v_{\varepsilon_\delta,\tau}(t))-F_r(v_{\varepsilon_\delta,\tau}(0))\leq h(t,\tau,v_{\varepsilon_\delta,\tau})+l(t,\tau,v_{\varepsilon_\delta,\tau})+C_{34}\int_0^t F_r(v_{\varepsilon_\delta,\tau})\,d\theta \text{ for a.e. } t. \tag{4.2.103}$$

Since $v_{\varepsilon_\delta,\tau}\to v_{\varepsilon_\delta}$ as $\tau\to 0$. $h(t,\tau,v_{\varepsilon_\delta,\tau})\to 0$ and $l(t,\tau,v_{\varepsilon_\delta,\tau})$ as $\tau\to 0$ for a.e. t. $F_r(v_{\varepsilon_\delta,\tau})$ is continuous (cf. lemma 4.1.1.2.4), i.e., $F_r(v_{\varepsilon_\delta,\tau})\to F_r(v_{\varepsilon_\delta})$ as $\tau\to 0$. Taking the limit on both sides of (4.2.103) as $\tau\to 0$, we get

$$F_r(v_{\varepsilon_\delta}(t))-F_r(v_{\varepsilon_\delta}(0))\leq C_{34}\int_0^t F_r(v_{\varepsilon_\delta})\,d\theta \qquad \text{for a.e. } t,$$

i.e.,

$$F_r(v_{\varepsilon_\delta}(t))\leq F_r(v_{\varepsilon_\delta}(0))+C_{34}\int_0^t F_r(v_{\varepsilon_\delta})\,d\theta \qquad \text{for a.e. } t.$$

Gronwall's inequality gives

$$F_r(v_{\varepsilon_\delta}(t))\leq e^{C_{34}t}F_r(v_{\varepsilon_\delta}(0)) \qquad \text{for a.e. } t. \tag{4.2.104}$$

where C_{34} is independent of ε, δ, λ and t. This establishes the inequality (4.2.78). ♦

Now we use the theorem 4.2.1.3.5 to obtain L^r - and L^∞ - estimates of the solution v_{ε_δ}. Let

$$C_{36}(r):=C_{36}:=\left[I_2\sup_{k,\varepsilon,\delta>0}\ \operatorname*{ess\,sup}_{0\leq t\leq T}|\Omega|\ C_{13}\,e^{C_{34}t}\left(1+\left(I_2^{\frac{1}{2}}\,|||v_0|||_{L^\infty(\Omega_\varepsilon^p)^{I_2}}\right)^{r(1+\alpha)}\right)\right]^{\frac{1}{r}} \tag{4.2.105}$$

and

$$C_{37}:=\sup_{\varepsilon,\delta>0}\left[1+\left(I_2^{\frac{1}{2}}\,|||v_0|||_{L^\infty(\Omega_\varepsilon^p)^{I_2}}\right)^{1+\alpha}\right]. \tag{4.2.106}$$

Corollary 4.2.1.3.7. *Let $\hat{u}_{\varepsilon_\delta}\in\mathcal{F}_\varepsilon^u$ be fixed. For any arbitrary solution $v_{\varepsilon_\delta}\in\mathcal{G}_\varepsilon^v$ of $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$ the following estimates hold true:*

$$\sup_{\varepsilon,\delta>0}|||v_{\varepsilon_\delta}(t)|||_{L^r(\Omega_\varepsilon^p)^{I_2}}\leq C_{36}<\infty \qquad \textit{for all } r \textit{ and for a.e. } t, \tag{4.2.107}$$

and

$$\sup_{\varepsilon,\delta>0}|||v_{\varepsilon_\delta}(t)|||_{L^\infty(\Omega_\varepsilon^p)^{I_2}}\leq C_{37}<\infty \qquad \textit{for a.e. } t. \tag{4.2.108}$$

Proof. Given $r \in \mathbb{N}\,(r \geq 2)$ and for the problem $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$, $v_{\varepsilon_\delta}(0,x) = \lambda v_0(x)$. From inequality (4.2.104), we have

$$F_r(v_{\varepsilon_\delta}(t)) \leq e^{C_{34}t} F_r(v_{\varepsilon_\delta}(0)) \qquad \text{for a.e. } t.$$

A straightforward application of Gronwall's inequality and arguments similar to the proof of corollary 4.1.1.2.8 yield the desired results. ♦

Corollary 4.2.1.3.8. *Let the assumptions (4.2.1)-(4.2.10), $\hat{u}_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$, $0 \leq \lambda \leq 1$ and $r \in \mathbb{N}$ be satisfied. Then there exists a constant C independent of $\hat{u}_{\varepsilon_\delta}$, v_{ε_δ}, ε, λ and t such that any arbitrary solution $v_{\varepsilon_\delta} \in \mathcal{G}^v_\varepsilon$ of the problem $(\bar{P}^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$ satisfies*

$$|||v_{\varepsilon_\delta}|||_{\mathcal{G}^v_\varepsilon} \leq C. \tag{4.2.109}$$

Proof. For $p > n+2$, $\hat{u}_{\varepsilon_\delta} \in L^\infty((0,T)\times\Omega^p_\varepsilon)^{I_1}$. Note that $v_{\varepsilon_{\delta_k}}$ satisfies the estimates (4.2.107) and (4.2.108). The abstract formulation of the problem (4.2.71)-(4.2.75) is given by

$$\frac{\partial v_{\varepsilon_\delta}}{\partial t} + A v_{\varepsilon_\delta} = f_{bound}(v_{\varepsilon_\delta}) + f(v_{\varepsilon_\delta}), \tag{4.2.110}$$

$$v_{\varepsilon_\delta}(0,x) = v_0(x), \tag{4.2.111}$$

where the operator A is defined as in remark 4.1.1.1.1 with maximal regularity on $[H^{1,q}(\Omega^p_\varepsilon)^*]^{I_2}$, $f(v_{\varepsilon_\delta}) = \lambda S_2 \bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}) + \lambda\,\kappa\, v_{\varepsilon_\delta}$ and $f_{bound}(v_{\varepsilon_\delta}) = -\vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_\delta} + Q_{\partial\Omega_{in}}(v_{\varepsilon_\delta}) + R_{\Gamma_\varepsilon}(-\lambda\frac{\partial w_{\varepsilon_\delta}}{\partial t})$. Choosing r sufficiently large in (4.2.107) and application of Hölder's inequality imply that $f \in L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_2}$. Since from lemma 4.2.1.3.1 $Q_{\partial\Omega_{in}}(v_{\varepsilon_\delta}) \in L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_2}$, by lemma 4.2.1.3.2 $R_{\Gamma_\varepsilon}(-\lambda\frac{\partial w_{\varepsilon_\delta}}{\partial t}) \in L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_2}$ and $-\vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_\delta} \in L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_2}$, the term f_{bound} is in $L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_2}$. Moreover from (4.2.3), we have $v_0 \in [(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}]^{I_2}$. Therefore from theorem 3.3.1 there exists a unique $v_{\varepsilon_\delta} \in \mathcal{G}^v_\varepsilon$ such that

$$|||v_{\varepsilon_\delta}|||_{\mathcal{G}^v_\varepsilon} \leq C, \tag{4.2.112}$$

where C is independent of λ and v_{ε_δ}. ♦

Lemma 4.2.1.3.9. *The operator Z_1 is compact and continuous.*

Proof. We will only show the compactness of Z_1 as the continuity follows analogously. Let $\hat{u}_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$ be fixed. Let $\{\hat{v}_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ be a bounded sequence in $\mathcal{G}^v_\varepsilon$. For $p > n+2$, $\mathcal{G}^v_\varepsilon \hookrightarrow\hookrightarrow L^\infty((0,T)\times\Omega^p_\varepsilon)^{I_2}$. Then up to a subsequence (still denoted by same symbol) $\{\hat{v}_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ is strongly convergent in $L^\infty((0,T)\times\Omega^p_\varepsilon)^{I_2}$. Therefore the r.h.s of the PDE

$$\frac{\partial v_{\varepsilon_{\delta_n}}}{\partial t} - \nabla(D\nabla v_{\varepsilon_{\delta_n}} - \vec{q}_\varepsilon v_{\varepsilon_{\delta_n}}) + \kappa v_{\varepsilon_{\delta_n}} = S_2 \bar{R}(\hat{u}_{\varepsilon_\delta}, \hat{v}_{\varepsilon_{\delta_n}}) + \kappa\,\hat{v}_{\varepsilon_{\delta_n}}$$

is strongly convergent in $L^p((0,T);L^p(\Omega^p_\varepsilon))^{I_2}$, i.e., in $L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_2}$. Thus by theorem 3.3.1, the sequence $\{v_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ is strongly convergent in $\mathcal{G}^v_\varepsilon$. ♦

Proof of theorem 4.2.1.3.3. The compactness and continuity of the operator Z_1 is shown in the lemma 4.2.1.3.9 and the corollary 4.2.1.3.8 gives the estimate (4.2.70). By Schaefer's fixed point theorem the operator Z_1 has a fixed point, i.e., the problem $(\bar{P}^{2+}_{\varepsilon_{\delta_M}})$ has a solution. This solution is also a solution of $(\bar{P}^{2+}_{\varepsilon_\delta})$. ♦

4.2.1.4 Existence of the Global Solution of the Complete Problem $(P^{2+}_{\varepsilon_\delta})$

Theorem 4.2.1.4.1. *There exists a positive weak solution* $(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}, w_{\varepsilon_\delta}) \in \mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$ *of the following problem:*

$$
\begin{aligned}
\frac{\partial u_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta}) &= S_1\bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) && \text{in } (0,T)\times\Omega^p_\varepsilon,\\
u_{\varepsilon_\delta}(0,x) &= u_0(x) && \text{in } \Omega^p_\varepsilon,\\
-(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta})\cdot\vec{n} &= d && \text{on } (0,T)\times\partial\Omega_{in},\\
-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{out},\\
-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} &= 0 && \text{on } (0,T)\times\Gamma_\varepsilon,\\
\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) &= S_2\bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) && \text{in } (0,T)\times\Omega^p_\varepsilon,\\
v_{\varepsilon_\delta}(0,x) &= v_0(x) && \text{in } \Omega^p_\varepsilon,\\
-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{in},\\
-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} &= 0 && \text{on } (0,T)\times\partial\Omega_{out},\\
-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} &= \varepsilon\frac{\partial w_\varepsilon}{\partial t} && \text{on } (0,T)\times\Gamma_\varepsilon,\\
\frac{\partial w_{\varepsilon_\delta}}{\partial t} &= -k_d\psi_\delta(w_{\varepsilon_\delta}) && \text{on } (0,T)\times\Gamma_\varepsilon,\\
w_{\varepsilon_\delta}(0,x) &= w_0(x) && \text{on } \Gamma_\varepsilon.
\end{aligned}
$$

The positivity of the solution has already been shown in lemma 4.2.1.1.2. To prove the existence of the global solution of the problem $(P^{2+}_{\varepsilon_\delta})$, we employ the similar techniques which we used to solve $(\bar{P}^{2+}_{\varepsilon_\delta})$. Here also the basic ingredients are Schaefer's fixed point theorem, a Lyapunov functional and theorem 3.3.1. The Lyapunov functional is defined in the following way: Let $\bar{\mu}^0 \in \mathbb{R}^{I_1}$ be the solution of

$$S_1^T\bar{\mu}^0 = -\log K, \tag{4.2.113}$$

where $K \in \mathbb{R}^J$ is the vector of equilibrium constants $K_j = \frac{k^f_j}{k^b_j}$. Due to (4.2.4), (4.2.113) has a solution. The function $g_i : \mathbb{R}^+_0 \to \mathbb{R}$ is defined by $g_i(u_{\varepsilon_{\delta_i}}) := \left(\bar{\mu}^0_i - 1 + \log u_{\varepsilon_{\delta_i}}\right) u_{\varepsilon_{\delta_i}} + e^{(1-\bar{\mu}^0_i)}$. We define the functions g, f_r and F_r in the similar way as we did in section 4.1.1.2 and their relevant properties hold good (see propositions 4.1.1.2.1 and 4.1.1.2.2). Now for technical reasons, we modify the right hand side of the first PDE in $(P^{2+}_{\varepsilon_\delta})$, i.e., for any $\kappa > 0$, we get

$$\frac{\partial u_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta}) + \kappa u_{\varepsilon_\delta} = S_1\bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) + \kappa u_{\varepsilon_\delta} \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.114}$$

$$u_{\varepsilon_\delta}(0,x) = u_0(x) \quad \text{in } \Omega^p_\varepsilon, \tag{4.2.115}$$

$$-(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta})\cdot\vec{n} = d \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.116}$$

$$-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.117}$$

$$-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.118}$$

$$\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) = S_2\bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.119}$$

$$v_{\varepsilon_\delta}(0,x) = v_0(x) \quad \text{in } \Omega^p_\varepsilon, \tag{4.2.120}$$

$$-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.121}$$

$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.122}$$

$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\frac{\partial w_\varepsilon}{\partial t} \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.123}$$

$$\frac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d \psi_\delta(w_{\varepsilon_\delta}) \qquad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.124}$$
$$w_{\varepsilon_\delta}(0,x) = w_0(x) \qquad \text{on } \Gamma_\varepsilon. \tag{4.2.125}$$

Let us denote this problem by $(P^{2+}_{\varepsilon_{\delta_M}})$. We define the fixed point operator $Z_2 : \mathcal{F}^u_\varepsilon \to \mathcal{F}^u_\varepsilon$ via $Z_2(\hat{u}_{\varepsilon_\delta}) := u_{\varepsilon_\delta}$, where u_{ε_δ} is the solution of the following linear problem

$$\frac{\partial u_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta}) + \kappa u_{\varepsilon_\delta} = S_1 \bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}) + \kappa \hat{u}_{\varepsilon_\delta} \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.126}$$
$$u_{\varepsilon_\delta}(0,x) = u_0(x) \qquad \text{in } \Omega^p_\varepsilon, \tag{4.2.127}$$
$$-(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta})\cdot\vec{n} = d \qquad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.128}$$
$$-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.129}$$
$$-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.130}$$

where $v_{\varepsilon_\delta} \in \mathcal{G}^v_\varepsilon$ is the solution of the problem

$$\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) = S_2 \bar{R}(\hat{u}_{\varepsilon_\delta}, v_{\varepsilon_\delta}) \qquad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.131}$$
$$v_{\varepsilon_\delta}(0,x) = v_0(x) \qquad \text{in } \Omega^p_\varepsilon, \tag{4.2.132}$$
$$-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.133}$$
$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.134}$$
$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\frac{\partial w_{\varepsilon_\delta}}{\partial t} \qquad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.135}$$

and $w_{\varepsilon_\delta} \in \mathcal{H}^w_\varepsilon$ is the solution of the problem

$$\frac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d \psi_\delta(w_{\varepsilon_\delta}) \qquad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.136}$$
$$w_{\varepsilon_\delta}(0,x) = w_0(x) \qquad \text{on } \Gamma_\varepsilon. \tag{4.2.137}$$

Note that for fixed $\hat{u}_{\varepsilon_\delta}$ the problem (4.2.131)-(4.2.137) has a solution and satisfies the estimates (4.2.107)-(4.2.108). The operator Z_2 is well-defined (can be verified as in remark 4.2.1.3.4). Now in order to apply the Schaefer's fixed point theorem, we show the following two condition:

(i) The operator Z_2 is continuous and compact.

(ii) The set $\{u_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon | \exists\lambda \in [0,1] : u_{\varepsilon_\delta} = \lambda Z_2(u_{\varepsilon_\delta})\}$ is bounded, i.e., there exists a constant $C > 0$ independent of u_{ε_δ} and λ such that any arbitrary solution $u_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$ of the equation

$$u_{\varepsilon_\delta} = \lambda Z_2(u_{\varepsilon_\delta}) \tag{4.2.138}$$

satisfies

$$|||u_{\varepsilon_\delta}|||_{\mathcal{F}^u_\varepsilon} \le C. \tag{4.2.139}$$

Combining (4.2.126)-(4.2.137) and (4.2.138), we obtain

$$\frac{\partial u_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta}) + \kappa u_{\varepsilon_\delta} = \lambda S_1 \bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) + \lambda\kappa u_{\varepsilon_\delta} \quad \text{in } (0,T)\times\Omega_\varepsilon^p, \tag{4.2.140}$$

$$u_{\varepsilon_\delta}(0,x) = \lambda u_0(x) \quad \text{in } \Omega_\varepsilon^p, \tag{4.2.141}$$

$$-(D\nabla u_{\varepsilon_\delta} - \vec{q}_\varepsilon u_{\varepsilon_\delta})\cdot\vec{n} = \lambda d \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.142}$$

$$-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.143}$$

$$-D\nabla u_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.144}$$

where $v_{\varepsilon_\delta} \in \mathcal{G}_\varepsilon^v$ is the solution of the problem

$$\frac{\partial v_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta}) = S_2\bar{R}(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}) \quad \text{in } (0,T)\times\Omega_\varepsilon^p, \tag{4.2.145}$$

$$v_{\varepsilon_\delta}(0,x) = v_0(x) \quad \text{in } \Omega_\varepsilon^p, \tag{4.2.146}$$

$$-(D\nabla v_{\varepsilon_\delta} - \vec{q}_\varepsilon v_{\varepsilon_\delta})\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.147}$$

$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.148}$$

$$-D\nabla v_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\frac{\partial w_{\varepsilon_\delta}}{\partial t} \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.149}$$

and $w_{\varepsilon_\delta} \in \mathcal{H}_\varepsilon^w$ is the solution of the problem

$$\frac{\partial w_{\varepsilon_\delta}}{\partial t} = -k_d\psi_\delta(w_{\varepsilon_\delta}) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.150}$$

$$w_{\varepsilon_\delta}(0,x) = w_0(x) \quad \text{on } \Gamma_\varepsilon. \tag{4.2.151}$$

Let us call the problem (4.2.140)-(4.2.151) as $(P^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$. The inequality (4.2.139) is the consequence of the following three results:

Lemma 4.2.1.4.2. *Let $p > n+2$, $0 \le \lambda \le 1$ and $r \in \mathbb{N}$ $(r \ge 2)$. Assume that $u_{\varepsilon_\delta} \in \mathcal{F}_\varepsilon^u$ is a solution of $(P^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$ and for $\tau > 0$,*

$$u_{\varepsilon_\delta,\tau} := u_{\varepsilon_\delta} + \tau.$$

Then the following inequality holds:

$$\int_0^t \left\langle \frac{\partial u_{\varepsilon_\delta,\tau}}{\partial\theta}, \partial f_r(u_{\varepsilon_\delta,\tau})\right\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^{I_1}\times H^{1,q}(\Omega_\varepsilon^p)^{I_1}} d\theta$$
$$\le h(t,\tau,u_{\varepsilon_\delta,\tau}) + l(t,\tau,u_{\varepsilon_\delta,\tau}) + C_{38}\int_0^t F_r(u_{\varepsilon_\delta,\tau})\,d\theta \qquad \text{for a.e. } t,$$

where $h(t,\tau,u_{\varepsilon_\delta,\tau})$ and $l(t,\tau,u_{\varepsilon_\delta,\tau})$ tend to zero as $\tau \to 0$ for a.e. t, and C_{38} is independent of ε, δ, λ and t.

Proof. Note that for $u_{\varepsilon_\delta} \in \mathcal{F}_\varepsilon^u$, the problem (4.2.145)-(4.2.151) has a solution $(v_{\varepsilon_\delta}, w_{\varepsilon_\delta}) \in \mathcal{G}_\varepsilon^v \times \mathcal{H}_\varepsilon^w$ with estimates (4.2.107)-(4.2.108). We use $\partial f_r(u_{\varepsilon_\delta}) \in L^q((0,T);H^{1,q}(\Omega_\varepsilon^p))^{I_1}$ as the test function in the weak formulation of (4.2.140). Replicating the steps of lemma 4.2.1.3.6 and use of (4.2.10) will finish the proof. ♦

Theorem 4.2.1.4.3. *Let $r \in \mathbb{N}$ $(r \ge 2)$, $0 \le t \le T$ and $0 \le \lambda \le 1$. Suppose that $u_{\varepsilon_\delta} \in \mathcal{F}_\varepsilon^u$ is a solution of $(P^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$. Then the following inequality holds good:*

$$F_r(u_{\varepsilon_\delta}(t)) \le e^{C_{38}t}\,F_r(u_{\varepsilon_\delta}(0)) \qquad \text{for a.e. } t, \tag{4.2.152}$$

where C_{38} is independent of ε, δ, λ and t.

Proof. The proof follows from lemma 4.2.1.4.2 by using arguments similar to the proof of theorem 4.2.1.3.5. ♦

Corollary 4.2.1.4.4. *For any arbitrary solution $u_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$ of $(P^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$ the following estimates hold true:*

$$|||u_{\varepsilon_\delta}(t)|||_{L^r(\Omega^p_\varepsilon)^{I_1}} \leq C_{39} < \infty \quad \text{for all } r \text{ and for a.e. } t, \tag{4.2.153}$$

and

$$|||u_{\varepsilon_\delta}(t)|||_{L^\infty(\Omega^p_\varepsilon)^{I_1}} \leq C_{40} < \infty \quad \text{for a.e. } t, \tag{4.2.154}$$

where C_{39} and C_{40} are independent of ε, δ, λ and t.

Proof. By using arguments from the proof of corollary 4.2.1.3.7 in (4.2.152) yield the proof. ♦

Corollary 4.2.1.4.5. *Let the assumptions (4.2.1)-(4.2.10), $0 \leq \lambda \leq 1$ and $r \in \mathbb{N}$ be satisfied. Then there exists a constant C independent of u_{ε_δ}, ε, δ, λ and t such that any arbitrary solution $u_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$ of the problem $(P^{2+}_{\varepsilon_{\delta_{\lambda_M}}})$ satisfies*

$$|||u_{\varepsilon_\delta}|||_{\mathcal{F}^u_\varepsilon} \leq C. \tag{4.2.155}$$

Proof. The proof is analogous to the proof of corollary 4.2.1.3.8. ♦

Lemma 4.2.1.4.5. *The operator Z_2 is compact and continuous.*

Proof. Here we shall prove only the compactness of Z_2. Let $\{\hat{u}_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ be a bounded sequence in $\mathcal{F}^u_\varepsilon$. The proof will be done if we can show that up to a subsequence the r.h.s of the PDE

$$\frac{\partial u_{\varepsilon_{\delta_n}}}{\partial t} - \nabla\cdot(\nabla u_{\varepsilon_{\delta_n}} - \vec{q}_\varepsilon u_{\varepsilon_{\delta_n}}) + \kappa u_{\varepsilon_{\delta_n}} = S_1 \bar{R}(\hat{u}_{\varepsilon_{\delta_n}}, v_{\varepsilon_{\delta_n}}) + \kappa \hat{u}_{\varepsilon_{\delta_n}} \tag{4.2.156}$$

is strongly convergent in $L^p((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_1}$, where $v_{\varepsilon_{\delta_n}} \in \mathcal{G}^v_\varepsilon$ is the solution of the problem

$$\frac{\partial v_{\varepsilon_{\delta_n}}}{\partial t} - \nabla\cdot(D\nabla v_{\varepsilon_{\delta_n}} - \vec{q}_\varepsilon v_{\varepsilon_{\delta_n}}) = S_2 \bar{R}(\hat{u}_{\varepsilon_{\delta_n}}, v_{\varepsilon_{\delta_n}}) \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.157}$$

$$v_{\varepsilon_{\delta_n}}(0,x) = v_0(x) \quad \text{in } \Omega^p_\varepsilon, \tag{4.2.158}$$

$$-(D\nabla v_{\varepsilon_{\delta_n}} - \vec{q}_\varepsilon v_{\varepsilon_{\delta_n}})\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.159}$$

$$-D\nabla v_{\varepsilon_{\delta_n}}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.160}$$

$$-D\nabla v_{\varepsilon_{\delta_n}}\cdot\vec{n} = \varepsilon\frac{\partial w_{\varepsilon_{\delta_n}}}{\partial t} \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.161}$$

with estimate (4.2.109) and $w_{\varepsilon_{\delta_n}} \in \mathcal{H}^w_\varepsilon$ is the solution of the problem

$$\frac{\partial w_{\varepsilon_{\delta_n}}}{\partial t} = -k_d \psi_\delta(w_{\varepsilon_{\delta_n}}) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.162}$$

$$w_{\varepsilon_{\delta_n}}(0,x) = w_0(x) \quad \text{on } \Gamma_\varepsilon. \tag{4.2.163}$$

Thus the sequence $\{v_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ is bounded in $\mathcal{G}^v_\varepsilon$. Since $\mathcal{F}^u_\varepsilon$, $\mathcal{G}^v_\varepsilon \hookrightarrow\hookrightarrow L^\infty((0,T)\times\Omega^p_\varepsilon)$, up to a subsequence (still denoted by same symbol), $\{\hat{u}_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ and $\{v_{\varepsilon_{\delta_n}}\}_{n=1}^\infty$ are strongly convergent in $L^\infty((0,T)\times\Omega^p_\varepsilon)$ and this yields the strong convergence of the r.h.s of the PDE (4.2.156) in $L^q((0,T);H^{1,q}(\Omega^p_\varepsilon)^*)^{I_1}$. ♦

Proof of theorem 4.2.1.4.1. The corollary 4.2.1.4.5 and lemma 4.2.1.4.5 show that the conditions of Schaefer's fixed point theorem are satisfied. Hence there exists at least one fixed point of Z_2, i.e., the problem $(P^{2^+}_{\varepsilon_{\delta_M}})$ has a solution $(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}, w_{\varepsilon_\delta}) \in \mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$. The solution of $(P^{2^+}_{\varepsilon_{\delta_M}})$ is also a solution of $(P^{2^+}_{\varepsilon_\delta})$. ♦

Proof of theorem 4.2.1.1.1. Since in lemma 4.2.1.1.2 we have shown that the solution of $(P^{2^+}_{\varepsilon_\delta})$ is nonnegative, the solution also solves the problem $(P^2_{\varepsilon_\delta})$. In the next section, we prove the uniqueness of the solution of $(P^2_{\varepsilon_\delta})$. ♦

4.2.1.5 Uniqueness of the Solution of the Problem $(P^2_{\varepsilon_\delta})$

Theorem 4.2.1.5.1. *There exists a unique positive global solution $(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}, w_{\varepsilon_\delta}) \in \mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$ of the problem $(P^2_{\varepsilon_\delta})$.*

Proof. On the contrary, let us assume that $(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}, w_{\varepsilon_\delta,1})$ and $(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2}, w_{\varepsilon_\delta,2})$ be the solutions of the problem $(P^2_{\varepsilon_\delta})$. Set $\bar{u}_{\varepsilon_\delta} := u_{\varepsilon_\delta,1} - u_{\varepsilon_\delta,2}$, $\bar{v}_{\varepsilon_\delta} := v_{\varepsilon_\delta,1} - v_{\varepsilon_\delta,2}$ and $\bar{w}_{\varepsilon_\delta} := w_{\varepsilon_\delta,1} - w_{\varepsilon_\delta,2}$. Let the systems satisfied by $(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}, w_{\varepsilon_\delta,1})$ and $(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2}, w_{\varepsilon_\delta,2})$ be denoted by $(P^2_{\varepsilon_{\delta_1}})$ and $(P^2_{\varepsilon_{\delta_2}})$ respectively. Substracting the systems of equations of $(P^2_{\varepsilon_{\delta_1}})$ and $(P^2_{\varepsilon_{\delta_2}})$, we get

$$\frac{\partial \bar{u}_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla\bar{u}_{\varepsilon_\delta} - \vec{q}_\varepsilon \bar{u}_{\varepsilon_\delta}) = S_1 R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}) - S_1 R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2}) \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.164}$$

$$\bar{u}_{\varepsilon_\delta}(0) = 0 \quad \text{in } \Omega^p_\varepsilon, \tag{4.2.165}$$

$$-(D\nabla\bar{u}_{\varepsilon_\delta} - \vec{q}_\varepsilon \bar{u}_{\varepsilon_\delta})\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.166}$$

$$-D\nabla\bar{u}_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.167}$$

$$-D\nabla\bar{u}_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.168}$$

$$\frac{\partial \bar{v}_{\varepsilon_\delta}}{\partial t} - \nabla\cdot(D\nabla\bar{v}_{\varepsilon_\delta} - \vec{q}_\varepsilon \bar{v}_{\varepsilon_\delta}) = S_2 R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}) - S_2 R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2}) \quad \text{in } (0,T)\times\Omega^p_\varepsilon, \tag{4.2.169}$$

$$\bar{v}_{\varepsilon_\delta}(0) = 0 \quad \text{in } \Omega^p_\varepsilon, \tag{4.2.170}$$

$$-(D\nabla\bar{v}_{\varepsilon_\delta} - \vec{q}_\varepsilon \bar{v}_{\varepsilon_\delta})\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.171}$$

$$-D\nabla\bar{v}_{\varepsilon_\delta}\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.172}$$

$$-D\nabla\bar{v}_{\varepsilon_\delta}\cdot\vec{n} = \varepsilon\frac{\partial \bar{w}_{\varepsilon_\delta}}{\partial t} \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.173}$$

$$\frac{\partial \bar{w}_{\varepsilon_\delta}}{\partial t} = -k_d(\psi_\delta(w_{\varepsilon_\delta,1}) - \psi_\delta(w_{\varepsilon_\delta,2})) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{4.2.174}$$

$$\bar{w}_{\varepsilon_\delta}(0) = 0 \quad \text{on } \Gamma_\varepsilon. \tag{4.2.175}$$

(i) **Uniqueness of the ODE (4.2.42)-(4.2.43):** Here $\bar{w}_{\varepsilon_\delta} := w_{\varepsilon_\delta,1} - w_{\varepsilon_\delta,2} \in H^{1,p}((0,T); L^p(\Gamma_\varepsilon))^{I_2}$. We multiply the ODE (4.2.174) with $\bar{w}_{\varepsilon_\delta}$ and integrate over $(0,t)\times\Gamma_\varepsilon$. Employement of the *Lipschitz continuity* of ψ_δ and a straightforward application of Gronwall's inequality yield the desired result.
(ii) **Uniqueness of the PDE (4.2.32)-(4.2.36) and (4.2.37)-(4.2.41):** Testing the equation

(4.2.169) with $\bar{v}_{\varepsilon_\delta}$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{I_2}\int_0^t \frac{d}{d\theta}\left|\left|\bar{v}_{\varepsilon_{\delta_k}}(t)\right|\right|^2_{L^2(\Omega_\varepsilon^p)} d\theta + D\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\nabla \bar{v}_{\varepsilon_{\delta_k}}\right|^2 dx\, d\theta\\
&\quad + \sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} \vec{q}_\varepsilon \nabla \bar{v}_{\varepsilon_{\delta_k}} \bar{v}_{\varepsilon_{\delta_k}}\, dx\, d\theta\\
&\qquad - \sum_{k=1}^{I_2}\int_0^t\int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n}\left|\bar{v}_{\varepsilon_{\delta_k}}\right|^2 ds\, d\theta + \sum_{k=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\varepsilon\frac{\partial \bar{w}_{\varepsilon_{\delta_k}}}{\partial t}\bar{v}_{\varepsilon_{\delta_k}}\, d\sigma_x\, d\theta\\
&\qquad\quad = \sum_{k=1}^{I_2}\int_0^t\left\langle S_2R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1})_k - S_2R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})_k, v_{\varepsilon_{\delta_k},1} - v_{\varepsilon_{\delta_k},2}\right\rangle d\theta,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{I_2}\int_0^t \frac{d}{d\theta}\left|\bar{v}_{\varepsilon_{\delta_k}}(t)\right|^2_{L^2(\Omega_\varepsilon^p)} d\theta + D\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\nabla \bar{v}_{\varepsilon_{\delta_k}}\right|^2 dx\, d\theta\\
&\quad = \underbrace{\sum_{k=1}^{I_2}\int_0^t\int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n}\left|\bar{v}_{\varepsilon_{\delta_k}}\right|^2 ds\, d\theta - \sum_{k=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\varepsilon\frac{\partial \bar{w}_{\varepsilon_{\delta_k}}}{\partial t}\bar{v}_{\varepsilon_{\delta_k}}\, d\sigma_x\, d\theta}_{=:I_{bound}}\\
&\qquad \underbrace{- \sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} \vec{q}_\varepsilon \nabla \bar{v}_{\varepsilon_{\delta_k}} \bar{v}_{\varepsilon_{\delta_k}}\, dx\, d\theta}_{=:I_{advec}}\\
&\qquad\quad \underbrace{+ \sum_{k=1}^{I_2}\int_0^t\left\langle S_2R(u_{\varepsilon_\delta,1}, v_{\varepsilon_{\delta_k},1})_k - S_2R(u_{\varepsilon_\delta,2}, v_{\varepsilon_{\delta_k},2})_k, v_{\varepsilon_{\delta_k},1} - v_{\varepsilon_{\delta_k},2}\right\rangle d\theta}_{=:I_{reac}}. \qquad (4.2.176)
\end{aligned}
$$

We simplify the boundary, advective and reaction terms separately. We start with the advective term.

$$
\begin{aligned}
I_{advec} &= -\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} \vec{q}_\varepsilon \nabla \bar{v}_{\varepsilon_{\delta_k}} \bar{v}_{\varepsilon_{\delta_k}}\, dx\, d\theta\\
&\leq \sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p} |\vec{q}_\varepsilon|\left|\nabla \bar{v}_{\varepsilon_{\delta_k}}\right|\left|\bar{v}_{\varepsilon_{\delta_k}}\right| dx\, d\theta\\
&\leq Q\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\nabla \bar{v}_{\varepsilon_{\delta_k}}\right|\left|\bar{v}_{\varepsilon_{\delta_k}}\right| dx\, d\theta, \text{ where } Q = ||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega_\varepsilon^p)}\\
&\underbrace{\leq}_{\substack{\text{Young's}\\ \text{inequality}}} \frac{2Q^2}{D}\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\bar{v}_{\varepsilon_{\delta_k}}\right|^2 dx\, d\theta + \frac{D}{8}\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\nabla \bar{v}_{\varepsilon_{\delta_k}}\right|^2 dx\, d\theta\\
&\leq C\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\bar{v}_{\varepsilon_{\delta_k}}\right|^2 dx\, d\theta + \frac{D}{8}\sum_{k=1}^{I_2}\int_0^t\int_{\Omega_\varepsilon^p}\left|\nabla \bar{v}_{\varepsilon_{\delta_k}}\right|^2 dx\, d\theta. \qquad (4.2.177)
\end{aligned}
$$

Next we simplify the boundary term.

$$
I_{bound} = \sum_{k=1}^{I_2}\int_0^t\int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n}\left|\bar{v}_{\varepsilon_{\delta_k}}\right|^2 ds\, d\theta - \sum_{k=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\varepsilon\frac{\partial \bar{w}_{\varepsilon_{\delta_k}}}{\partial t}\bar{v}_{\varepsilon_{\delta_k}}\, d\sigma_x\, d\theta.
$$

By part (i), $w_{\varepsilon_\delta,1}(t,x) = w_{\varepsilon_\delta,2}(t,x)$ for a.e. t and x. This implies that the boundary term on Γ_ε vanishes. On $\partial\Omega_{in}$, $\vec{q}_\varepsilon \cdot \vec{n} \leq 0$. Thus the integrand on $\partial\Omega_{in}$ is nonpositive. Therefore

$$I_{bound} \leq 0. \tag{4.2.178}$$

Finally, we simplify the reaction term.

$$\begin{aligned}
I_{reac} &= \sum_{k=1}^{I_2} \int_0^t \left\langle S_2 R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1})_k - S_2 R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})_k, v_{\varepsilon_{\delta_k},1} - v_{\varepsilon_{\delta_k},2} \right\rangle d\theta \\
&\leq \frac{1}{2} \sum_{k=1}^{I_2} \int_0^t \left[||S_2 R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1})_k - S_2 R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})_k||^2_{L^2(\Omega_\varepsilon^p)} + \left|\left| v_{\varepsilon_{\delta_k},1} - v_{\varepsilon_{\delta_k},2} \right|\right|^2_{L^2(\Omega_\varepsilon^p)} \right] d\theta.
\end{aligned} \tag{4.2.179}$$

Note that

$$\begin{aligned}
&||S_2 R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}) - S_2 R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})||^2_{L^2(\Omega_\varepsilon^p)} \\
&\qquad \leq \int_{\Omega_\varepsilon^p} \left(\sum_{j=1}^{J} |\nu_{kj}| \, |R_j(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}) - R_j(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})| \right)^2 dx.
\end{aligned}$$

Expanding the term $R_j(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}) - R_j(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})$, we will obtain two terms in which each term contains a factor of the type $u_{\varepsilon_{\delta_l},1} - u_{\varepsilon_{\delta_l},2}$ and $v_{\varepsilon_{\delta_m},1} - v_{\varepsilon_{\delta_m},2}$ whereas all the other factors are in $L^\infty((0,T) \times \Omega_\varepsilon^p)$. Therefore we obtain

$$\begin{aligned}
&||S_2 R(u_{\varepsilon_\delta,1}, v_{\varepsilon_\delta,1}) - S_2 R(u_{\varepsilon_\delta,2}, v_{\varepsilon_\delta,2})||^2_{L^2(\Omega_\varepsilon^p)} \\
&\qquad \leq \hat{C} \left[\sum_{i=1}^{I_1} \int_{\Omega_\varepsilon^p} \left| u_{\varepsilon_{\delta_i},1} - u_{\varepsilon_{\delta_i},2} \right|^2 dx + \sum_{k=1}^{I_2} \int_{\Omega_\varepsilon^p} \left| v_{\varepsilon_{\delta_k},1} - v_{\varepsilon_{\delta_k},2} \right|^2 dx \right].
\end{aligned} \tag{4.2.180}$$

Combining (4.2.176), (4.2.177), (4.2.178), (4.2.179) and (4.2.180), we obtain

$$\begin{aligned}
&\frac{1}{2} \sum_{k=1}^{I_2} \int_0^t \frac{d}{d\theta} \left|\left| \bar{v}_{\varepsilon_{\delta_k}}(t) \right|\right|^2_{L^2(\Omega_\varepsilon^p)} d\theta + D \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_\varepsilon^p} \left| \nabla \bar{v}_{\varepsilon_{\delta_k}} \right|^2 dx \, d\theta \\
&= I_{bound} + I_{advec} + I_{reac} \\
&\leq 0 + C \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_\varepsilon^p} \left| \bar{v}_{\varepsilon_{\delta_k}} \right|^2 dx \, d\theta + \frac{D}{8} \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_\varepsilon^p} \left| \nabla \bar{v}_{\varepsilon_{\delta_k}} \right|^2 dx \, d\theta \\
&\qquad + C \left[\int_0^t \int_{\Omega_\varepsilon^p} \sum_{i=1}^{I_1} \left| \bar{u}_{\varepsilon_{\delta_i}} \right|^2 dx \, d\theta + \int_0^t \int_{\Omega_\varepsilon^p} \sum_{k=1}^{I_2} \left| \bar{v}_{\varepsilon_{\delta_k}} \right|^2 dx \, d\theta \right] \\
\Longrightarrow &\frac{1}{2} \sum_{k=1}^{I_2} \int_0^t \frac{d}{d\theta} \left|\left| \bar{v}_{\varepsilon_{\delta_k}}(t) \right|\right|^2_{L^2(\Omega_\varepsilon^p)} d\theta \leq \bar{C}_1 \left(\sum_{k=1}^{I_2} \int_0^t \int_{\Omega_\varepsilon^p} \left| \bar{v}_{\varepsilon_{\delta_k}} \right|^2 dx \, d\theta + \sum_{i=1}^{I_1} \int_0^t \int_{\Omega_\varepsilon^p} \left| \bar{u}_{\varepsilon_{\delta_i}} \right|^2 dx \, d\theta \right).
\end{aligned} \tag{4.2.181}$$

Now we test the equation (4.2.164) by $\bar{u}_{\varepsilon_\delta}$ and proceed in the similar fashion as above, we obtain an inequality like (4.2.181) as

$$\frac{1}{2} \sum_{i=1}^{I_1} \int_0^t \frac{d}{d\theta} \left|\left| \bar{u}_{\varepsilon_{\delta_i}}(t) \right|\right|^2_{L^2(\Omega_\varepsilon^p)} d\theta \leq \bar{C}_2 \left(\sum_{k=1}^{I_2} \int_0^t \int_{\Omega_\varepsilon^p} \left| \bar{v}_{\varepsilon_{\delta_k}} \right|^2 dx \, d\theta + \sum_{i=1}^{I_1} \int_0^t \int_{\Omega_\varepsilon^p} \left| \bar{u}_{\varepsilon_{\delta_i}} \right|^2 dx \, d\theta \right). \tag{4.2.182}$$

Adding (4.2.181) and (4.2.182), we get

$$\frac{1}{2}\int_0^t \frac{d}{d\theta}\left(||\bar{u}_{\varepsilon\delta}||^2_{[L^2(\Omega^p_\varepsilon)]^{I_1}} + ||\bar{v}_{\varepsilon\delta}||^2_{[L^2(\Omega^p_\varepsilon)]^{I_2}}\right) d\theta \leq \bar{C}_3\int_0^t \left(||\bar{u}_{\varepsilon\delta}||^2_{[L^2(\Omega^p_\varepsilon)]^{I_1}} + ||\bar{v}_{\varepsilon\delta}||^2_{[L^2(\Omega^p_\varepsilon)]^{I_2}}\right) d\theta,$$

i.e.,

$$||\bar{u}_{\varepsilon\delta}(t)||^2_{[L^2(\Omega^p_\varepsilon)]^{I_1}} + ||\bar{v}_{\varepsilon\delta}(t)||^2_{[L^2(\Omega^p_\varepsilon)]^{I_2}} \leq 2\bar{C}_3\int_0^t \left(||\bar{u}_{\varepsilon\delta}(\theta)||^2_{[L^2(\Omega^p_\varepsilon)]^{I_1}} + ||\bar{v}_{\varepsilon\delta}(\theta)||^2_{[L^2(\Omega^p_\varepsilon)]^{I_2}}\right) d\theta.$$

Since $u_{\varepsilon\delta_i,1}(0) = u_{\varepsilon\delta_i,2}(0)$ and $v_{\varepsilon\delta_k,1}(0) = v_{\varepsilon\delta_k,2}(0)$ for all i and k, therefore Gronwall's inequality gives

$$\begin{aligned} ||\bar{u}_{\varepsilon\delta}(t)||^2_{[L^2(\Omega^p_\varepsilon)]^{I_1}} + ||\bar{v}_{\varepsilon\delta}(t)||^2_{[L^2(\Omega^p_\varepsilon)]^{I_2}} &= 0 \text{ for a.e. } t\\ \Longrightarrow u_{\varepsilon\delta,1} = u_{\varepsilon\delta,2} &\text{ and } v_{\varepsilon\delta,1} = v_{\varepsilon\delta,2}. \end{aligned}$$

Hence the problem $(P^2_{\varepsilon_\delta})$ has a unique positive global weak solution in $\mathcal{F}^u_\varepsilon \times \mathcal{G}^v_\varepsilon \times \mathcal{H}^w_\varepsilon$. ♦

4.2.2 Homogenization of the Problem $(P^2_{\varepsilon_\delta})$

Now keeping δ fixed, we upscale model M2 from the micro to the macro scale. The basic ingredients are the *a-priori estimates* of the solutions, *two-scale convergence* and *periodic unfolding*. We begin with the equations for immobile species.

4.2.2.1 A-priori Estimates of the Solution of the Problem (4.2.12)-(4.2.23)

4.2.2.1.1 A-priori Estimates of the Solution of the ODE (4.2.22)-(4.2.23)

Theorem 4.2.2.1.1.1. *Let w_{ε_δ} be the solution of the problem (4.2.22)-(4.2.23), then it satisfies the following estimates:*

$$\begin{aligned} &\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^2 d\sigma_x\,dt + \varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^2 d\sigma_x\,dt\\ &\quad + \varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x\,dt + \varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt \leq C_{41} < \infty, \end{aligned} \tag{4.2.183}$$

where C_{41} is independent of ε, δ, m and t.

Proof. The proof consists of several steps.
(a) Multiplying both sides of (4.2.22) by w_{ε_δ} and integrating over $(0,t)\times\Gamma_\varepsilon$, we obtain

$$\int_0^t\int_{\Gamma\varepsilon}\left\langle\frac{\partial w_{\varepsilon_\delta}(\theta,x)}{\partial\theta}, w_{\varepsilon_\delta}(\theta,x)\right\rangle_{I_2} d\sigma_x\,d\theta = -k_d\int_0^t\int_{\Gamma_\varepsilon}\langle\psi_\delta(w_{\varepsilon_\delta}(\theta,x)), w_{\varepsilon_\delta}(\theta,x)\rangle_{I_2}\,d\sigma_x\,d\theta,$$

i.e.,

$$\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\frac{\partial w_{\varepsilon\delta_m}(\theta,x)}{\partial\theta}w_{\varepsilon\delta_m}(\theta,x)\,d\sigma_x\,d\theta = -k_d\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\psi_\delta(w_{\varepsilon\delta_m}(\theta,x))w_{\varepsilon\delta_m}(\theta,x)\,d\sigma_x\,d\theta,$$

i.e.,

$$\frac{1}{2}\sum_{m=1}^{I_2}\int_0^t\frac{\partial}{\partial\theta}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(\theta,x)\right|^2 d\sigma_x\,d\theta \leq k_d\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\left|\psi_\delta(w_{\varepsilon\delta_m}(\theta,x))\right|\left|w_{\varepsilon\delta_m}(\theta,x)\right| d\sigma_x\,d\theta,$$

i.e.,

$$\frac{1}{2}\sum_{m=1}^{I_2}\int_0^t \frac{\partial}{\partial\theta}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta$$
$$\leq \frac{1}{2}\sum_{m=1}^{I_2}\left[\int_0^t\int_{\Gamma_\varepsilon} k_d^2\,d\sigma_x\,d\theta + \int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta\right], \because \left|\psi_\delta(w_{\varepsilon_{\delta_m}}(\theta,x))\right| \leq 1,$$

i.e.,

$$\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(t,x)\right|^2 d\sigma_x$$
$$\leq \sum_{m=1}^{I_2}\left[\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(0,x)\right|^2 d\sigma_x + k_d^2\int_0^t\int_{\Gamma_\varepsilon} d\sigma_x\,d\theta + \int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta\right]$$
$$\leq \sum_{m=1}^{I_2}\left[\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(0,x)\right|^2 d\sigma_x + \frac{k_d^2}{\varepsilon}\,T\,|\Gamma|\,|\Omega| + \int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta\right],$$

i.e.,

$$\varepsilon\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(t,x)\right|^2 d\sigma_x$$
$$\leq \varepsilon\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(0,x)\right|^2 d\sigma_x + k_d^2\,T\,|\Gamma|\,|\Omega|\,I_2 + \varepsilon\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta$$
$$\leq |\Gamma|\,|\Omega|^{1-\frac{2}{p}}\sum_{m=1}^{I_2}||w_{0_m}||^2_{L^p(\Omega)} + k_d^2\,T\,|\Gamma|\,|\Omega|\,I_2 + \varepsilon\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta$$
$$\leq |\Gamma|\,|\Omega|^{1-\frac{2}{p}}\sum_{m=1}^{I_2}\sup_{\varepsilon>0}||w_{0_m}||^2_{L^p(\Omega)} + k_d^2\,T\,|\Gamma|\,|\Omega|\,I_2 + \varepsilon\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\,d\theta,$$

i.e.,

$$\varepsilon\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(t,x)\right|^2 d\sigma_x \leq \bar{C} + \varepsilon\int_0^t\left(\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(\theta,x)\right|^2 d\sigma_x\right)d\theta,$$

where $\bar{C}$ $(:= |\Gamma|\,|\Omega|^{1-\frac{2}{p}}\sum_{m=1}^{I_2}\sup_{\varepsilon>0}||w_{0_m}||^2_{L^p(\Omega)} + k_d^2\,T\,|\Gamma|\,|\Omega|\,I_2 < \infty)$ is a constant independent of ε.[36] Application of Gronwall's inequality gives

$$\varepsilon\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(t,x)\right|^2 d\sigma_x \leq \bar{C}(1+te^t),$$

i.e.,

$$\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon_{\delta_m}}(t,x)\right|^2 d\sigma_x\,dt \leq \bar{C}\int_0^T(1+te^t)\,dt,$$

[36]Note that $\sum_{m=1}^{I_2}\sup_{\varepsilon>0}||w_{0_m}||^2_{L^p(\Omega)} < \infty$ by (4.2.5).

i.e.,

$$\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^2 d\sigma_x\,dt\le C_{42}, \tag{4.2.184}$$

where C_{42} $(:=\bar{C}\int_0^T(1+te^t)\,dt)$ is independent of ε, δ, m and t.

(b) Now multiplying the equation (4.2.22) by $\frac{\partial w_{\varepsilon\delta}}{\partial t}$ and integrating over $(0,T)\times\Gamma_\varepsilon$, we get

$$\int_0^T\int_{\Gamma_\varepsilon}\left\langle\frac{\partial w_{\varepsilon\delta}(t,x)}{\partial t},\frac{\partial w_{\varepsilon\delta}(t,x)}{\partial t}\right\rangle_{I_2} d\sigma_x\,dt=-k_d\int_0^T\int_{\Gamma_\varepsilon}\left\langle\psi_\delta(w_{\varepsilon\delta}(t,x)),\frac{\partial w_{\varepsilon\delta}(t,x)}{\partial t}\right\rangle_{I_2} d\sigma_x\,dt,$$

i.e.,

$$\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^2 d\sigma_x\,dt\le\frac{1}{2}\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left[k_d^2\left|\psi_\delta(w_{\varepsilon\delta_m}(t,x))\right|^2+\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^2\right] d\sigma_x\,dt,$$

i.e.,

$$\frac{1}{2}\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^2 d\sigma_x\,dt\le\sum_{m=1}^{I_2}\frac{k_d^2}{2}\int_0^T\int_{\Gamma_\varepsilon} d\sigma_x\,dt,\quad\text{since }\left|\psi_\delta(w_{\varepsilon\delta_m}(t,x))\right|\le 1,$$

i.e.,

$$\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^2 d\sigma_x\,dt\le k_d^2 I_2\,|\Gamma|\,\frac{|\Omega|}{|Y|}T=:C_{43}, \tag{4.2.185}$$

where C_{43} is independent of ε, δ, m and t.

(c) Again multiplying the m-*th* ODE of (4.2.22) by $\frac{\partial w_{\varepsilon\delta_m}}{\partial t}\left|\frac{\partial w_{\varepsilon\delta_m}}{\partial t}\right|^{p-2}$ and integrating over $(0,T)\times\Gamma_\varepsilon$, we get

$$\begin{aligned}\int_0^T\int_{\Gamma_\varepsilon}&\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^{p-2} d\sigma_x\,dt\\&=-\int_0^T\int_{\Gamma_\varepsilon}k_d\psi_\delta(w_{\varepsilon\delta_m}(t,x))\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^{p-2} d\sigma_x\,dt,\end{aligned}$$

i.e.,

$$\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt\le\int_0^T\int_{\Gamma_\varepsilon}|k_d|\left|\psi_\delta(w_{\varepsilon\delta_m}(t,x))\right|\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^{p-1} d\sigma_x\,dt,$$

i.e.,

$$\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt\le\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^{p-1}k_d\,d\sigma_x\,dt,$$

i.e.,

$$\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt\underbrace{\le}_{\text{Young's inequality}}\int_0^T\int_{\Gamma_\varepsilon}\left[\frac{p-1}{p}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p+\frac{1}{p}k_d^p\right] d\sigma_x\,dt,$$

i.e.,

$$\frac{1}{p}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt \le \frac{k_d^p}{p}\int_0^T\int_{\Gamma_\varepsilon} d\sigma_x\,dt,$$

i.e.,

$$\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt \le k_d^p\frac{|\Omega|}{|Y|}\,|\Gamma|,$$

i.e.,

$$\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt \le I_2\,k_d^p\frac{|\Omega|}{|Y|}\,|\Gamma| =: C_{44}, \tag{4.2.186}$$

where C_{44} is independent of ε, δ, m and t.

(d) Multiplying both sides of the m-*th* ODE of (4.2.22) by $w_{\varepsilon\delta_m}\left|w_{\varepsilon\delta_m}\right|^{p-2}$ and integrating

$$\begin{aligned}
&\int_0^t\int_{\Gamma_\varepsilon} w_{\varepsilon\delta_m}(t,x)\left|w_{\varepsilon\delta_m}(t,x)\right|^{p-2}\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\,d\sigma_x\,dt\\
&\qquad = -k_d\int_0^t\int_{\Gamma_\varepsilon} w_{\varepsilon\delta_m}(t,x)\left|w_{\varepsilon\delta_m}(t,x)\right|^{p-2}\psi_\delta(w_{\varepsilon\delta_m}(t,x))\,d\sigma_x\,dt,
\end{aligned}$$

i.e.,

$$\int_0^t\int_{\Gamma_\varepsilon}\frac{1}{p}\frac{\partial}{\partial t}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x\,dt \le \int_0^t\int_{\Gamma_\varepsilon}\left[\frac{p-1}{p}\left|w_{\varepsilon\delta_m}(t,x)\right|^p+\frac{k_d^p}{p}\right]d\sigma_x\,dt,$$

i.e.,

$$\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x \le \int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(0,x)\right|^p d\sigma_x+\int_0^t\int_{\Gamma_\varepsilon}\left[(p-1)\left|w_{\varepsilon\delta_m}(t,x)\right|^p+k_d^p\right]d\sigma_x\,dt,$$

i.e.,

$$\begin{aligned}
&\varepsilon\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x\\
&\qquad\le \varepsilon\sum_{m=1}^{I_2}\int_{\Gamma_\varepsilon}|w_{0_m}(x)|^p\,d\sigma_x+T\,I_2\,|\Gamma|\,|\Omega|\,k_d^p+(p-1)\,\varepsilon\sum_{m=1}^{I_2}\int_0^t\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x\,dt.
\end{aligned}$$

A straightforward application of Gronwall's inequality and steps similar to part (a) will imply

$$\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x\,dt \le C_{45}, \tag{4.2.187}$$

where C_{45} is independent of ε, δ, m and t.

Therefore adding (4.2.184), (4.2.185), (4.2.186) and (4.2.187), we obtain

$$\begin{aligned}
&\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^2 d\sigma_x\,dt+\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^2 d\sigma_x\,dt\\
&\qquad+\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|w_{\varepsilon\delta_m}(t,x)\right|^p d\sigma_x\,dt+\varepsilon\sum_{m=1}^{I_2}\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon\delta_m}(t,x)}{\partial t}\right|^p d\sigma_x\,dt\\
&\le C_{42}+C_{43}+C_{45}+C_{44}\\
&= C_{41},
\end{aligned}$$

where C_{41} $(:= C_{42}+C_{43}+C_{44}+C_{45})$ is independent of ε, δ, m and t. ♦

4.2.2.1.2 Extension of the Solution of the PDE (4.2.17)-(4.2.21)

Theorem 4.2.2.1.2.1. *There exists an extension of the solution v_{ε_δ} of the problem (4.2.17)-(4.2.21) to all of $(0,T)\times\Omega$ such that*

$$|||v_{\varepsilon_\delta}|||_{L^r((0,T);L^r(\Omega))^{I_2}} + |||v_{\varepsilon_\delta}|||_{L^\infty((0,T);L^\infty(\Omega))^{I_2}} + |||\nabla v_{\varepsilon_\delta}|||_{L^2((0,T);L^2(\Omega))^{I_2}} \leq C_{46}, \quad (4.2.188)$$

where C_{46} is independent of ε, δ, k and t but depends on r.

The proof of the above theorem resides on the following lemma:

Lemma 4.2.2.1.2.2. *Let $p>n+2$ and $r\in\mathbb{N}$. Suppose that v_{ε_δ} is the solution of the problem (4.2.17)-(4.2.21), then it satisfies the following estimate*

$$|||v_{\varepsilon_\delta}|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^{I_2}} + |||v_{\varepsilon_\delta}|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^{I_2}} + |||\nabla v_{\varepsilon_\delta}|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^{I_2}} \leq C_{47}, \quad (4.2.189)$$

where C_{47} is independent of ε, δ, k and t but depends on r.

Proof. The proof of this lemma consists of several intermediate steps.
(a) Following the arguments of lemma 4.1.2.1.2, we obtain

$$|||v_{\varepsilon_\delta}|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^{I_2}} \leq C_{48} \quad (4.2.190)$$

and

$$|||v_{\varepsilon_\delta}|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^{I_2}} \leq C_{49}, \quad (4.2.191)$$

where C_{48} and C_{49} are independent of ε, δ, k and t.
(b) Testing the $k-th$ PDE of the system of equation (4.2.17) by $v_{\varepsilon_{\delta_k}}$ and integrating over $(0,T)$, we get

$$\int_0^T \left\langle \frac{\partial v_{\varepsilon_{\delta_k}}}{\partial t}, v_{\varepsilon_{\delta_k}} \right\rangle dt - \int_0^T \left\langle \nabla D \nabla v_{\varepsilon_{\delta_k}}, v_{\varepsilon_{\delta_k}} \right\rangle dt + \int_0^T \int_{\Omega^p_\varepsilon} \vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_{\delta_k}} v_{\varepsilon_{\delta_k}} \, dx\, dt$$
$$= \int_0^T \left\langle (S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}))_k, v_{\varepsilon_{\delta_k}} \right\rangle dt,$$

i.e.,

$$\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt} \left\| v_{\varepsilon_{\delta_k}} \right\|^2_{L^2(\Omega^p_\varepsilon)} dt + D\int_0^T \int_{\Omega^p_\varepsilon} \left|\nabla v_{\varepsilon_{\delta_k}}\right|^2 dx\, dt \\
&= \int_0^T \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon_{\delta_k}} v_{\varepsilon_{\delta_k}} \, ds\, dt - \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon_{\delta_m}}}{\partial t} v_{\varepsilon_{\delta_k}} \, d\sigma_x\, dt \\
&\quad - \int_0^T \int_{\Omega^p_\varepsilon} \vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_{\delta_k}} v_{\varepsilon_{\delta_k}} \, dx\, dt + \int_0^T \left\langle (S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}))_k, v_{\varepsilon_{\delta_k}} \right\rangle dt \\
&=: I_{bound} + I_{advec} + I_{reac},
\end{aligned} \quad (4.2.192)$$

where

$$I_{bound} := \int_0^T \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon_{\delta_k}} v_{\varepsilon_{\delta_k}} \, ds\, dt - \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon_{\delta_m}}}{\partial t} v_{\varepsilon_{\delta_k}} \, d\sigma_x\, dt, \quad (4.2.193)$$

$$I_{advec} := -\int_0^T \int_{\Omega^p_\varepsilon} \vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_{\delta_k}} v_{\varepsilon_{\delta_k}} \, dx\, dt, \quad (4.2.194)$$

$$I_{reac} := \int_0^T \left\langle (S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}))_k, v_{\varepsilon_{\delta_k}} \right\rangle dt. \quad (4.2.195)$$

We simplify the terms I_{bound}, I_{advec} and I_{reac} one by one.

$$\begin{aligned}
&I_{bound}\\
&= \int_0^T \int_{\partial\Omega_{in}} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon_{\delta_k}} v_{\varepsilon_{\delta_k}} \, ds\, dt - \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon_{\delta_m}}}{\partial t} v_{\varepsilon_{\delta_k}} \, d\sigma_x \, dt\\
&\le \int_0^T \int_{\partial\Omega_{in}} ||\vec{q}_\varepsilon \cdot \vec{n}||^2_{L^\infty((0,T)\times\partial\Omega_{in})} \left|v_{\varepsilon_{\delta_k}}\right|^2 ds\, dt + \varepsilon k_d \int_0^T \int_{\Gamma_\varepsilon} \left|v_{\varepsilon_{\delta_k}}\right| d\sigma_x\, dt, \because \left|\psi_\delta(w_{\varepsilon_{\delta_m}})\right| \le 1\\
&\le ||\vec{q}_\varepsilon \cdot \vec{n}||^2_{L^\infty((0,T)\times\partial\Omega_{in})} \int_0^T \int_{\partial\Omega} \left|v_{\varepsilon_{\delta_k}}\right|^2 ds\, dt + \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \left[\tau_1 \left|v_{\varepsilon_{\delta_k}}\right|^2 + c(\tau_1) k_d^2\right] d\sigma_x\, dt.
\end{aligned} \tag{4.2.196}$$

Note that

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega} \left|v_{\varepsilon_{\delta_k}}\right|^2 ds\, dt\\
&\le c\left(\left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|_{L^2((0,T)\times\Omega_\varepsilon^p)} \left|\left|v_{\varepsilon_{\delta_k}}\right|\right|_{L^2((0,T)\times\Omega_\varepsilon^p)} + \left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right), \text{ by theorem 3.4.3.2}\\
&\le c\left(\tau_2 \left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + c(\tau_2) \left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + \left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right)
\end{aligned}$$

$$= c\left(\tau_2 \left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + \hat{c}(\tau_2) \left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right), \text{ where } \hat{c}(\tau_2) = c(\tau_2) + 1 \tag{4.2.197}$$

and from theorem 3.4.1.3,

$$\begin{aligned}
\varepsilon \int_0^T \int_{\Gamma_\varepsilon} \left|v_{\varepsilon_{\delta_k}}\right|^2 d\sigma_x\, dt &\le c\left(\left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + \varepsilon \left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right)\\
&\le c\left(\left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + \left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right),
\end{aligned} \tag{4.2.198}$$

since $0 < \varepsilon \ll 1$. Combining (4.2.196), (4.2.197) and (4.2.198), we get

$$\begin{aligned}
I_{bound} \le{}& ||\vec{q}_\varepsilon \cdot \vec{n}||^2_{L^\infty((0,T)\times\partial\Omega_{in})} c\left(\tau_2 \left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + \hat{c}(\tau_2) \left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right)\\
&+ c\tau_1 \left(\left|\left|v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)} + \left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega_\varepsilon^p)}\right) + \varepsilon c(\tau_1) k_d^2 \int_0^T \int_{\Gamma_\varepsilon} d\sigma_x\, dt.
\end{aligned} \tag{4.2.199}$$

Next,

$$\begin{aligned}
I_{advec} &\le \int_0^T \int_{\Omega_\varepsilon^p} \left|\vec{q}_\varepsilon \cdot \nabla v_{\varepsilon_{\delta_k}}\right| \left|v_{\varepsilon_{\delta_k}}\right| dx\, dt\\
&\le \tau_3 \int_0^T \int_{\Omega_\varepsilon^p} \left|\nabla v_{\varepsilon_{\delta_k}}\right|^2 dx\, dt + Q^2 c(\tau_3) \int_0^T \int_{\Omega_\varepsilon^p} \left|v_{\varepsilon_{\delta_k}}\right|^2 dx\, dt,
\end{aligned} \tag{4.2.200}$$

where $Q = ||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega_\varepsilon^p)}$.

$$I_{reac} \le \int_0^T \left|\left\langle (S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}))_k, v_{\varepsilon_{\delta_k}}\right\rangle\right| dt \quad \le \frac{1}{2} \int_0^T \int_{\Omega_\varepsilon^p} \left[|(S_2 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta}))_k|^2 + \left|v_{\varepsilon_{\delta_k}}\right|^2\right] dx\, dt. \tag{4.2.201}$$

Combining (4.2.192), (4.2.199), (4.2.200) and (4.2.201), we get

$$\frac{1}{2} \int_0^T \frac{d}{dt} \left|v_{\varepsilon_{\delta_k}}\right|^2_{L^2(\Omega_\varepsilon^p)} dt + D \int_{S\times\Omega_\varepsilon^p} \left|\nabla v_{\varepsilon_{\delta_k}}\right|^2 dx\, dt = I_{bound} + I_{advec} + I_{reac}$$

$$
\begin{aligned}
&\le \|\vec{q}_\varepsilon\cdot\vec{n}\|^2_{L^\infty((0,T)\times\partial\Omega_{in})}\, c\left(\tau_2\left\|\nabla v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}+\hat{c}(\tau_2)\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\right)\\
&\quad+c\tau_1\left(\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}+\left\|\nabla v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\right)+\varepsilon c(\tau_1)k_d^2\int_0^T\int_{\Gamma_\varepsilon} d\sigma_x\,dt\\
&\quad+\tau_3\int_0^T\int_{\Omega^p_\varepsilon}\left|\nabla v_{\varepsilon\delta_k}\right|^2dx\,dt+Q^2c(\tau_3)\int_0^T\int_{\Omega^p_\varepsilon}\left|v_{\varepsilon\delta_k}\right|^2dx\,dt,\\
&\quad+\frac{1}{2}\int_0^T\int_{\Omega^p_\varepsilon}\left[|(S_2R(u_{\varepsilon\delta},v_{\varepsilon\delta}))_k|^2+\left|v_{\varepsilon\delta_k}\right|^2\right]dx\,dt.
\end{aligned}
\tag{4.2.202}
$$

Choosing $\tau_1=\frac{D}{8c}$, $\tau_2=\frac{D}{8c\|\vec{q}_\varepsilon\cdot\vec{n}\|^2_{L^\infty((0,T)\times\partial\Omega_{in})}}$ and $\tau_3=\frac{D}{8}$, then (4.2.202) reduces to

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\frac{d}{dt}\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2(\Omega^p_\varepsilon)}dt+\frac{5D}{8}\int_0^T\int_{\Omega^p_\varepsilon}\left|\nabla v_{\varepsilon\delta_k}\right|^2dx\,dt\\
&\le \|\vec{q}_\varepsilon\cdot\vec{n}\|^2_{L^\infty((0,T)\times\partial\Omega_{in})}c\hat{c}(\tau_2)\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}+c\tau_1\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}+\varepsilon c(\tau_1)k_d^2\\
&\int_0^T\int_{\Gamma_\varepsilon}d\sigma_x\,dt+Q^2c(\tau_3)\int_0^T\int_{\Omega^p_\varepsilon}\left|v_{\varepsilon\delta_k}\right|^2dx\,dt+\frac{1}{2}\int_0^T\int_{\Omega^p_\varepsilon}\left[|(S_2R(u_{\varepsilon\delta},v_{\varepsilon\delta}))_k|^2+\left|v_{\varepsilon\delta_k}\right|^2\right]dx\,dt,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{1}{2}\left\|v_{\varepsilon\delta_k}(T)\right\|^2_{L^2(\Omega^p_\varepsilon)}dt+\frac{5D}{8}\int_0^T\int_{\Omega^p_\varepsilon}\left|\nabla v_{\varepsilon\delta_k}\right|^2dx\,dt\\
&\le\frac{1}{2}\|v_k(0)\|^2_{L^2(\Omega^p_\varepsilon)}+\|\vec{q}_\varepsilon\cdot\vec{n}\|^2_{L^\infty((0,T)\times\partial\Omega_{in})}c\hat{c}(\tau_2)\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}+c\tau_1\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\\
&\quad+c(\tau_1)k_d^2T|\Gamma|\frac{|\Omega|}{|Y|}+Q^2c(\tau_3)\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\\
&\qquad+\frac{1}{2}\left[\|(S_2R(u_{\varepsilon\delta},v_{\varepsilon\delta}))_k\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}+\left\|v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\right].
\end{aligned}
\tag{4.2.203}
$$

From the assumptions (4.2.3) and (4.2.5) it follows that $\sup_{\varepsilon>0}\|v_k(0)\|_{L^2(\Omega^p_\varepsilon)}<\infty$. Choosing r sufficiently large in the inequalities (4.2.107) and (4.2.153) and employment of the Hölder's inequality give $\sup_{\varepsilon>0}\|(S_2R(u_{\varepsilon\delta},v_{\varepsilon\delta}))_k\|_{L^2((0,T)\times\Omega^p_\varepsilon)}\le C_{50}<\infty$. Thus the whole r.h.s. of (4.2.203) is bounded by a constant independent of ε, δ, k and t, i.e.,

$$
\begin{aligned}
\left\|\nabla v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\le C_{51}\implies&\sum_{k=1}^{I_2}\left\|\nabla v_{\varepsilon\delta_k}\right\|^2_{L^2((0,T)\times\Omega^p_\varepsilon)}\le C_{51}I_2\\
\implies&\sup_{\varepsilon>0}|||\nabla v_{\varepsilon\delta}|||_{L^2((0,T)\times\Omega^p_\varepsilon)^{I_2}}\le C_{52},
\end{aligned}
\tag{4.2.204}
$$

where C_{52} $(:=(C_{51}I_2)^{\frac{1}{2}})$ is independent of ε, δ, k, and t but depends on r. Therefore adding (4.2.190),(4.2.191) and (4.2.204), we get

$$
\begin{aligned}
&|||v_{\varepsilon\delta}|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^{I_2}}+|||v_{\varepsilon\delta}|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^{I_2}}+|||\nabla v_{\varepsilon\delta}|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^{I_2}}\\
&\le C_{48}+C_{49}+C_{52}\\
&=C_{47},
\end{aligned}
\tag{4.2.205}
$$

where C_{47} $(:=C_{48}+C_{49}+C_{52})$ is independent of ε, δ, k and t but depends on r. ♦

Proof of theorem 4.2.2.1.2.1: The estimate (4.2.188) from lemma 4.2.2.1.2.2 and theorem 3.4.2.3 finish off the proof. ♦

4.2.2.1.3 Extension of the Solution of the PDE (4.2.12)-(4.2.16)

We can extend the solution $u_{\varepsilon_\delta} \in \mathcal{F}^u_\varepsilon$ of the problem (4.2.12)-(4.2.16) to all of $(0,T)\times\Omega$ as we did for v_{ε_δ} in section 4.2.2.1.2. For the extension we use the following two results:

Theorem 4.2.2.1.3.1. *There exists an extension of the solution u_{ε_δ} of the problem (4.2.12)-(4.2.16) to all of $(0,T)\times\Omega$ such that*

$$|||u_{\varepsilon\delta}|||_{L^r((0,T);L^r(\Omega))^{I_1}} + |||u_{\varepsilon\delta}|||_{L^\infty((0,T);L^\infty(\Omega))^{I_1}} + |||\nabla u_{\varepsilon\delta}|||_{L^2((0,T);L^2(\Omega))^{I_1}} \le C_{53}, \quad (4.2.206)$$

where C_{53} is independent of ε, δ, i and t but depends on r.

Lemma 4.2.2.1.3.2. *Let $p > n+2$ and $r \in \mathbb{N}$. Suppose that u_{ε_δ} is the solution of the problem (4.2.12)-(4.2.16), then it satisfies the following estimates*

$$|||u_{\varepsilon\delta}|||_{L^r((0,T);L^r(\Omega^p_\varepsilon))^{I_1}} + |||u_{\varepsilon\delta}|||_{L^\infty((0,T);L^\infty(\Omega^p_\varepsilon))^{I_1}} + |||\nabla u_{\varepsilon\delta}|||_{L^2((0,T);L^2(\Omega^p_\varepsilon))^{I_1}} \le C_{54}, \quad (4.2.207)$$

where C_{54} is independent of ε, δ, i and t but depends on r.

4.2.2.2 Convergence of the Micro Solution

4.2.2.2.1 Convergence of the Micro Solution of the Problem (4.2.22)-(4.2.26)

Theorem 4.2.2.2.1.1. *The solution, v_{ε_δ}, of the problem $(P^2_{\varepsilon_\delta})$ satisfies the following estimate:*

$$|||v_{\varepsilon\delta}|||_{L^\infty((0,T);L^2(\Omega))^{I_2}} + |||v_{\varepsilon\delta}|||_{L^2((0,T);H^{1,2}(\Omega))^{I_2}} + \left|\left|\left|\chi^\varepsilon \frac{\partial v_{\varepsilon\delta}}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}} \le C_{55}, \quad (4.2.208)$$

where C_{55} is independent of ε, δ, k and t but depends on r.

Proof. (a) The arguments similar to part (a) and (b) of theorem 4.1.2.2.1 yields

$$|||v_{\varepsilon\delta}|||_{L^\infty((0,T);L^2(\Omega))^{I_2}} \le C_{56} \quad (4.2.209)$$

and

$$|||v_{\varepsilon\delta}|||_{L^2((0,T);H^{1,2}(\Omega))^{I_2}} \le C_{57}, \quad (4.2.210)$$

where C_{56} and C_{57} are independent of ε, δ, k and t.

(b) Now let $\phi \in H^{1,2}_0(0,T)$ and $\psi \in H^{1,2}(\Omega)$. Then the weak formulation of the k-*th* PDE of the problem (4.2.17)-(4.2.21) is given by

$$\begin{aligned}
&\int_0^T \left\langle \chi\left(\frac{x}{\varepsilon}\right) \frac{\partial v_{\varepsilon_{\delta_k}}(t)}{\partial t}, \phi(t)\psi \right\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)} dt + \int_0^T \int_\Omega D\phi(t)\chi\left(\frac{x}{\varepsilon}\right) \nabla v_{\varepsilon_{\delta_k}}(t,x)\nabla\psi(x)\,dx\,dt \\
&- \int_0^T \int_{\partial\Omega_{in}} \vec{q}_\varepsilon\cdot\vec{n} v_{\varepsilon_{\delta_k}}(t,x)\phi(t)\psi(x)\,ds\,dt + \varepsilon\int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon_{\delta_k}}(t,x)}{\partial t}\phi(t)\psi(x)\,d\sigma_x\,dt \\
&+ \int_0^T \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) \vec{q}\cdot\nabla v_{\varepsilon_{\delta_k}}(t,x)\phi(t)\psi(x)\,dx\,dt \\
&= \int_0^T \left\langle \chi\left(\frac{x}{\varepsilon}\right) S_2R(u_{\varepsilon_\delta}(t), v_{\varepsilon_\delta}(t))_k, \phi(t)\psi \right\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)} dt,
\end{aligned}$$

i.e.,

$$\int_0^T \left\langle \chi\left(\frac{x}{\varepsilon}\right)\frac{\partial v_{\varepsilon_{\delta_k}}(t)}{\partial t}, \phi(t)\psi\right\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)} dt$$
$$\le \int_0^T\int_\Omega D\left|\chi\left(\frac{x}{\varepsilon}\right)\right|\left|\nabla v_{\varepsilon_{\delta_k}}(x)\right||\nabla\psi(x)||\phi(t)|\,dx\,dt$$
$$+||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega)}\int_0^T\int_\Omega\left|\nabla v_{\varepsilon_{\delta_k}}(t,x)\right||\phi(t)||\psi(x)|\,dx\,dt$$
$$+||\vec{q}_\varepsilon\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}\int_0^T\int_{\partial\Omega}\left|v_{\varepsilon_{\delta_k}}(t,x)\right||\psi(x)||\phi(t)|\,ds\,dt+\varepsilon$$
$$\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon_{\delta_k}}(t,x)}{\partial t}\right||\phi(t)||\psi(x)|\,d\sigma_x\,dt+\int_0^T\left\langle\chi\left(\frac{x}{\varepsilon}\right)S_2R(u_{\varepsilon_\delta}(t),v_{\varepsilon_\delta})_k(t),\phi(t)\psi\right\rangle_{L^2(\Omega)\times L^2(\Omega)} dt. \tag{4.2.211}$$

We estimate each term on the r.h.s. of (4.2.211) one by one. The first term can be estimated as

$$\int_0^T\int_\Omega D\left|\chi\left(\frac{x}{\varepsilon}\right)\right|\left|\nabla v_{\varepsilon_{\delta_k}}(t,x)\right||\nabla\psi(x)||\phi(t)|\,dx\,dt$$
$$+||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega)}\int_0^T\int_\Omega\left|\nabla v_{\varepsilon_{\delta_k}}(t,x)\right||\phi(t)||\psi(x)|\,dx\,dt$$
$$\le\frac{D}{2}\left[\int_0^T\int_\Omega\left|\nabla v_{\varepsilon_{\delta_k}}(t,x)\right|^2dx\,dt+\int_0^T\int_\Omega|\phi(t)|^2|\nabla\psi(x)|^2\,dx\,dt\right]$$
$$+\frac{||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega)}}{2}\int_0^T\int_\Omega\left[\left|\nabla v_{\varepsilon_{\delta_k}}(t,x)\right|^2+|\phi(t)|^2|\psi(x)|^2\right],\ \text{since}\ \left|\chi\left(\frac{x}{\varepsilon}\right)\right|\le 1$$
$$\le\frac{D}{2}\left[\left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega)}+||\phi||^2_{L^2(0,T)}||\nabla\psi||^2_{L^2(\Omega)}\right]$$
$$+\frac{||\vec{q}_\varepsilon||_{L^\infty((0,T)\times\Omega)}}{2}\left[\left|\left|\nabla v_{\varepsilon_{\delta_k}}\right|\right|^2_{L^2((0,T)\times\Omega)}+||\phi||^2_{L^2(0,T)}||\psi||^2_{L^2(\Omega)}\right]. \tag{4.2.212}$$

Again,[37]

$$\int_0^T\int_{\partial\Omega}\left|v_{\varepsilon_{\delta_k}}(t,x)\right||\psi(x)||\phi(t)|\,ds\,dt$$
$$\le\frac{1}{2}\int_0^T\int_{\partial\Omega}\left[\left|v_{\varepsilon_{\delta_k}}(t,x)\right|^2+|\psi(x)|^2|\phi(t)|^2\right]ds\,dt$$
$$\le\frac{C}{2}\int_0^T\left[\int_\Omega\left(\left|v_{\varepsilon_{\delta_k}}\right|^2+\left|\nabla v_{\varepsilon_{\delta_k}}\right|^2\right)dx+|\phi|^2\int_\Omega\left(|\psi|^2+|\nabla\psi|^2\right)dx\right]dt$$
$$=\frac{C}{2}\left[\int_0^T\int_\Omega\left|v_{\varepsilon_{\delta_k}}\right|^2+\left|\nabla v_{\varepsilon_{\delta_k}}\right|^2dx\,dt+\left(\int_0^T|\phi|^2\,dt\right)\left(\int_\Omega|\psi|^2+|\nabla\psi|^2\,dx\right)\right]. \tag{4.2.213}$$

For the third term,[38]

$$\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\left|\frac{\partial w_{\varepsilon_{\delta_k}}(t,x)}{\partial t}\right||\phi(t)||\psi(x)|\,d\sigma_x\,dt\le\frac{\varepsilon}{2}\int_0^T\int_{\Gamma_\varepsilon}\left[\left|\frac{\partial w_{\varepsilon_{\delta_k}}}{\partial t}\right|^2+|\phi(t)|^2|\psi(x)|^2\right]d\sigma_x\,dt$$
$$=\frac{1}{2}\left[\varepsilon\int_0^T\int_\Omega\left|\frac{\partial w_{\varepsilon_{\delta_k}}}{\partial t}\right|^2d\sigma_x\,dt+\int_0^T|\phi|^2\,dt\varepsilon\int_{\Gamma_\varepsilon}|\psi|^2\,d\sigma_x\right]$$

[37] We have used the boundary inequality (3.4.22).
[38] In this case, we used the inequality (3.4.5).

$$\leq \frac{1}{2}\left[\varepsilon\int_0^T\int_\Omega\left|\frac{\partial w_{\varepsilon\delta_k}}{\partial t}\right|^2 d\sigma_x\,dt+\int_0^T|\phi(t)|^2\,dt\,C\left(\int_{\Omega_\varepsilon^p}|\psi(x)|^2+\varepsilon^2\int_{\Omega_\varepsilon^p}|\nabla\psi(x)|^2\right)dx\right]$$
$$\leq \frac{1}{2}\left[\varepsilon\int_0^T\int_\Omega\left|\frac{\partial w_{\varepsilon\delta_k}}{\partial t}\right|^2 d\sigma_x\,dt+\int_0^T|\phi(t)|^2\,dt\,C\left(\int_{\Omega}|\psi(x)|^2+\int_{\Omega}|\nabla\psi(x)|^2\right)dx\right], \quad (4.2.214)$$

since $0<\varepsilon<<1$. Finally, the fourth term can be estimated as

$$\int_0^T\left\langle\chi\left(\frac{x}{\varepsilon}\right)S_2R(u_{\varepsilon_\delta},v_{\varepsilon_\delta})_k,\phi(t)\psi\right\rangle_{L^2(\Omega)\times L^2(\Omega)}dt$$
$$\leq\frac{1}{2}\int_0^T\int_\Omega\left[\left|\chi\left(\frac{x}{\varepsilon}\right)S_2R(u_{\varepsilon_\delta},v_{\varepsilon_\delta})_k\right|^2+|\phi(t)|^2\,|\psi(x)|^2\right]dx\,dt. \quad (4.2.215)$$

The inequalities (4.2.212), (4.2.213), (4.2.214) and (4.2.215) can be further estimated by (4.2.183) and (4.2.188). Following the similar steps as shown in the proof of theorem 4.1.2.2.1, we obtain

$$\begin{aligned}
&\left\|\chi^\varepsilon\frac{\partial v_{\varepsilon_{\delta_k}}}{\partial t}\right\|_{L^2((0,T);H^{1,2}(\Omega)^*)} && \leq C_{58}\\
\Longrightarrow\;&\sum_{k=1}^{I_2}\left\|\chi^\varepsilon\frac{\partial v_{\varepsilon_{\delta_k}}}{\partial t}\right\|^2_{L^2((0,T);H^{1,2}(\Omega)^*)} && \leq C_{58}^2 I_2\\
\Longrightarrow\;&\left|\left|\left|\chi^\varepsilon\frac{\partial v_{\varepsilon_\delta}}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}} && \leq C_{59} \quad (4.2.216)
\end{aligned}$$

where C_{59} $(:=(C_{58}^2 I_2)^{\frac{1}{2}})$ is independent of ε, δ, k and t but depends on r. Adding (4.2.209), (4.2.210) and (4.2.216), we get

$$|||v_{\varepsilon_\delta}|||_{L^\infty((0,T);L^2(\Omega))^{I_2}}+|||v_{\varepsilon_\delta}|||_{L^2((0,T);H^{1,2}(\Omega))^{I_2}}+\left|\left|\left|\chi^\varepsilon\frac{\partial v_{\varepsilon_\delta}}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}}$$
$$\leq C_{56}+C_{57}+C_{59}\;=:\;C_{55},$$

where C_{55} is independent of ε, δ, k and t but depends on r. ♦

Theorem 4.2.2.2.1.2. *Let $(v_{\varepsilon_\delta})_{\varepsilon>0}$ satisfies the estimates (4.2.188) and (4.2.208). Then there exists a function $v_\delta\in L^2((0,T);H^{1,2}(\Omega))^{I_2}$ and a function $v_\delta^1\in L^2((0,T)\times\Omega;H^{1,2}_{per}(Y)/\mathbb{R})^{I_2}$ such that up to a subsequence, still denoted by same subscript, the following convergence results hold:*

(i) $(v_{\varepsilon_\delta})_{\varepsilon>0}$ is weakly convergent to v_δ in $L^2((0,T);H^{1,2}(\Omega))^{I_2}$. (4.2.217)

(ii) $(v_{\varepsilon_\delta})_{\varepsilon>0}$ is strongly convergent to v_δ in $L^2((0,T);L^2(\Omega))^{I_2}$. (4.2.218)

(iii) $(v_{\varepsilon_\delta})_{\varepsilon>0}$ and $(\nabla_x v_{\varepsilon_\delta})_{\varepsilon>0}$ are two-scale convergent to v_δ and $\nabla_x v_\delta+\nabla_y v_\delta^1$ in the sense of (3.5.3). (4.2.219)

Proof. Given the *a-priori* estimates (4.2.188) and (4.2.208). With the help of theorem 3.5.13 and lemma 4.1.2.2.2, the proof follows like the proof of theorem 4.1.2.2.3. ♦

Corollary 4.2.2.2.1.3. *The limit function v_δ belongs to $L^\infty((0,T)\times\Omega\times Y)^{I_2}$.*[39]

[39] The function v_δ is independent of y.

Proof. Since $(v_{\varepsilon_\delta})_{\varepsilon>0}$ is strongly convergent in $L^2((0,T);L^2(\Omega))^{I_2}$, there exists a subsequence $(v_{\varepsilon'_\delta})_{\varepsilon'>0}$ which is pointwise convergent to v_δ almost everywhere in $(0,T)\times\Omega$, i.e,

$$\lim_{\varepsilon'\to 0} v_{\varepsilon'_\delta}(t,x) = v_\delta(t,x) \qquad \text{a.e.} \qquad (t,x)\in(0,T)\times\Omega.$$

By theorem 4.2.2.1.2.1, we have $\left|\left|v_{\varepsilon_{\delta_k}}\right|\right|_{L^\infty((0,T);L^\infty(\Omega))} \leq C_{46}$ for all k, where C_{46} is independent of ε and δ, therefore

$$\begin{aligned}
|v_{\delta_k}(t,x)|^2 \leq |v_\delta(t,x)|^2_{I_2} = \sum_{k=1}^{I_2} |v_{\delta_k}(t,x)|^2 &= \lim_{\varepsilon'\to 0} \sum_{k=1}^{I_2} \left| v_{\varepsilon'_{\delta_k}}(t,x)\right|^2 \\
&\leq \sum_{k=1}^{I_2} \overline{\lim_{\varepsilon'\to 0}} \ \operatorname*{ess\,sup}_{t\in(0,T)} \ \operatorname*{ess\,sup}_{x\in\Omega} \ \left| v_{\varepsilon'_{\delta_k}}(t,x)\right|^2 \\
&\leq \sum_{k=1}^{I_2} \overline{\lim_{\varepsilon'\to 0}} \ C_{46}^2 \\
&= C_{46}^2\ I_2 \text{ for a.e. } t \text{ and } x \\
\Longrightarrow \ \operatorname*{ess\,sup}_{t\in(0,T)} \ \operatorname*{ess\,sup}_{x\in\Omega} \ |v_{\delta_k}(t,x)|^2 &\leq C_{46}^2\ I_2 \\
\Longrightarrow \ \sup_{\delta>0} ||v_{\delta_k}||_{L^\infty((0,T);L^\infty(\Omega))} &\leq \left(C_{46}^2\ I_2\right)^{\frac{1}{2}} < \infty \quad \text{for all } k,
\end{aligned}$$

where $(C_{46}^2\ I_2)^{\frac{1}{2}}$ is independent of ε, δ and k. This gives

$$\begin{aligned}
|||v_\delta|||_{L^\infty((0,T)\times\Omega\times Y)^{I_2}} &= \max_{1\leq k\leq I_2} ||v_{\delta_k}||_{L^\infty((0,T)\times\Omega\times Y)} \\
&= \max_{1\leq k\leq I_2} \operatorname*{ess\,sup}_{(t,x,y)\in(0,T)\times\Omega\times Y} |v_{\delta_k}(t,x)| \\
&\leq \max_{1\leq k\leq I_2} \operatorname*{ess\,sup}_{y\in Y} \operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\Omega} |v_{\delta_k}(t,x)| \\
&\leq \max_{1\leq k\leq I_2} \operatorname*{ess\,sup}_{y\in Y} \operatorname*{ess\,sup}_{t\in(0,T)} \operatorname*{ess\,sup}_{x\in\Omega} |v_{\delta_k}(t,x)| \\
&\leq \operatorname*{ess\,sup}_{y\in Y} \left(C_{46}^2\ I_2\right)^{\frac{1}{2}} < \infty \\
\Longrightarrow \ \sup_{\delta>0} |||v_\delta|||_{L^\infty((0,T)\times\Omega\times Y)^{I_2}} &\leq \left(C_{46}^2\ I_2\right)^{\frac{1}{2}} < \infty
\end{aligned}$$

This completes the proof. ♦

4.2.2.2.2 Convergence of the Micro Solution of the Problem (4.2.12)-(4.2.16)

Next we state theorems concerning *weak*, *strong* and *two-scale* convergences of the sequence of functions (solution of (4.2.12)-(4.2.16)) u_{ε_δ}. These theorems can be proved in an anlogous way like the theorems 4.2.2.2.1.1 and 4.2.2.2.1.2.

Theorem 4.2.2.2.2.1. *The solution, u_{ε_δ}, of the problem $(P^2_{\varepsilon_\delta})$ satisfies the following estimate:*

$$|||u_{\varepsilon_\delta}|||_{L^\infty((0,T);L^2(\Omega))^{I_1}} + |||u_{\varepsilon_\delta}|||_{L^2((0,T);H^{1,2}(\Omega))^{I_1}} + \left|\left|\left|\chi^\varepsilon \frac{\partial u_{\varepsilon_\delta}}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}} \leq C_{60}, \tag{4.2.220}$$

where C_{60} is independent of ε, δ, i and t but depends on r.

Theorem 4.2.2.2.2.2. *Let $(u_{\varepsilon_\delta})_{\varepsilon>0}$ satisfies the estimates (4.2.206) and (4.2.220). Then there exists a function $u_\delta \in L^2((0,T);H^{1,2}(\Omega))^{I_1}$ and a function $u_\delta^1 \in L^2((0,T)\times\Omega;H^{1,2}_{per}(Y)/\mathbb{R})^{I_1}$ such that up to a subsequence, still denoted by same subscript, the following convergence results hold:*

(i) $(u_{\varepsilon_\delta})_{\varepsilon>0}$ *is weakly convergent to* u_δ *in* $L^2((0,T);H^{1,2}(\Omega))^{I_1}$. (4.2.221)

(ii) $(u_{\varepsilon_\delta})_{\varepsilon>0}$ *is strongly convergent to* u_δ *in* $L^2((0,T);L^2(\Omega))^{I_1}$. (4.2.222)

(iii) $(u_{\varepsilon_\delta})_{\varepsilon>0}$ *and* $(\nabla_x u_{\varepsilon_\delta})_{\varepsilon>0}$ *are two-scale convergent to* u_δ *and* $\nabla_x u_\delta + \nabla_y u_\delta^1$ *in the sense of* (3.5.3). (4.2.223)

Corollary 4.2.2.2.2.3. *The limit function u_δ belongs to $L^\infty((0,T)\times\Omega\times Y)^{I_1}$.*[40]

Proof. The proof is analogous to the proof of the theorem 4.2.2.2.1.3. ♦

Theorem 4.2.2.2.2.4. *The sequences $(S_1R(u_{\varepsilon_\delta},v_{\varepsilon_\delta}))_{\varepsilon>0}$ and $(S_2R(u_{\varepsilon_\delta},v_{\varepsilon_\delta}))_{\varepsilon>0}$ are strongly convergent to $S_1R(u_\delta,v_\delta)$ in $L^2((0,T);L^2(\Omega))^{I_1}$ and to $S_2R(u_\delta,v_\delta)$ in $L^2((0,T);L^2(\Omega))^{I_2}$ respectively.*

Proof. The proof is analogous to the proof of the theorem 4.1.2.2.6. The L^∞ - estimates in theorems 4.2.2.1.2.1, 4.2.2.1.3.1, 4.2.2.2.1.3 and 4.2.2.2.2.3, and the strong convergences in theorems 4.2.2.2.1.2 and 4.2.2.2.2.2 complete the proof. ♦

4.2.2.3 Passage to the Limit as $\varepsilon \to 0$

4.2.2.3.1 Homogenization of the ODE (4.2.22)-(4.2.23)

Using theorems 3.5.16 and 4.2.2.1.1.1, we can pass to the two-scale limit on the l.h.s. of (4.2.22) but due to the presence of nonlinear function on the r.h.s. of (4.2.22) one needs to pay special attention while passing the limit as $\varepsilon \to 0$. Here we take the help of periodic unfolding introduced in section 3.6 to pass to the limit in the nonlinear function $\psi_\delta(w_{\varepsilon_\delta})$.

Let $T_b^\varepsilon : L^2((0,T)\times\Gamma_\varepsilon) \to L^2((0,T)\times\Omega\times\Gamma)$ be the boundary unfolding operator defined as

$$T_b^\varepsilon w_{\varepsilon_{\delta_m}}(t,x,y) := w_{\varepsilon_{\delta_m}}(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y), \quad \text{for every } (t,x,y)\in(0,T)\times\Omega\times\Gamma. \tag{4.2.224}$$

Using the unfolding operator T_b^ε, we unfold the *m-th* ODE of (4.2.22)-(4.2.23). See that

$$\begin{aligned} T_b^\varepsilon\left(\frac{\partial w_{\varepsilon_{\delta_m}}}{\partial t}\right)(t,x,y) &= \frac{\partial}{\partial t}\left(w_{\varepsilon_{\delta_m}}\right)(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y) = \frac{\partial}{\partial t}\left(w_{\varepsilon_{\delta_m}}(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y)\right) \\ &= \frac{\partial}{\partial t}T_b^\varepsilon(w_{\varepsilon_{\delta_m}}(t,x,y)), \end{aligned}$$

$$\begin{aligned} T_b^\varepsilon\psi_\delta(w_{\varepsilon_{\delta_m}})(t,x,y) &= \psi_\delta(w_{\varepsilon_{\delta_m}})(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y) = \psi_\delta\left(w_{\varepsilon_{\delta_m}}(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y)\right) \\ &= \psi_\delta(T_\delta^\varepsilon w_{\varepsilon_{\delta_m}}(t,x,y)). \end{aligned}$$

Therefore the unfolded ODE is

$$\frac{\partial}{\partial t}T_b^\varepsilon w_{\varepsilon_{\delta_m}(t,x,y)} = -k_d\psi_\delta(T_b^\varepsilon w_{\varepsilon_{\delta_m}}(t,x,y)) \quad \text{in} \quad (0,T)\times\Omega\times\Gamma, \tag{4.2.225}$$

$$T_b^\varepsilon(w_{\varepsilon_{\delta_m}})(0,x,y) = T_b^\varepsilon w_{\varepsilon_{\delta_m}}(x,y) \quad \text{on} \quad \Omega\times\Gamma. \tag{4.2.226}$$

[40] The function u_δ is independent of y.

Lemma 4.2.2.3.1.1. *The sequence* $\left(T_b^{\varepsilon}(w_{\varepsilon_{\delta_m}})\right)_{\varepsilon>0}$ *is strongly convergent in* $L^2((0,T)\times\Omega\times\Gamma)$.

Proof. For $\nu,\mu\in\mathbb{N}$, let us consider the sequences $\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}\right)\right)_{\mu=1}^{\infty}$ and $\left(T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}\right)\right)_{\nu=1}^{\infty}$ which satisfies (4.2.225)-(4.2.226) such that

$$\frac{\partial}{\partial t}\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}\right)\right) = -k_d\psi_\delta\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}\right)\right) \quad \text{in} \quad (0,T)\times\Gamma_\varepsilon, \tag{4.2.227}$$

$$T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(0,x,y)\right) = T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(x,y)\right) \quad \text{on} \quad \Gamma_\varepsilon, \tag{4.2.228}$$

$$\frac{\partial}{\partial t}\left(T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}\right)\right) = -k_d\psi_\delta\left(T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}\right)\right) \quad \text{in} \quad (0,T)\times\Gamma_\varepsilon, \tag{4.2.229}$$

$$T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(0,x,y)\right) = T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(x,y)\right) \quad \text{on} \quad \Gamma_\varepsilon. \tag{4.2.230}$$

Substracting (4.2.227)-(4.2.229), we get

$$\begin{aligned}\frac{\partial}{\partial t}&\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}\right)\right)\\ &= -k_d\left[\psi_\delta\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}\right)\right) - \psi_\delta\left(T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}\right)\right)\right] \text{ in } (0,T)\times\Gamma_\varepsilon.\end{aligned} \tag{4.2.231}$$

Multiplying both sides by $T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}\right)$ and integrating over $(0,T)\times\Omega\times\Gamma$, we obtain[41]

$$\begin{aligned}\frac{1}{2}\int_0^t\int_\Omega\int_\Gamma &\frac{\partial}{\partial\theta}\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(\theta,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(\theta,x,y)\right)\right|^2 dx\,dy\,d\theta\\ &= -k_d\int_0^t\int_\Omega\int_\Gamma\left[\psi_\delta\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(\theta,x,y)\right)\right) - \psi_\delta\left(T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(\theta,x,y)\right)\right)\right]\\ &\qquad\left(T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(\theta,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(\theta,x,y)\right)\right) dx\,dy\,d\theta\\ &\le k_d K_{Lip}\int_0^t\int_\Omega\int_\Gamma\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(\theta,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(\theta,x,y)\right)\right|^2 dx\,dy\,d\theta,\end{aligned}$$

i.e.,

$$\begin{aligned}&\left|\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(t)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(t)\right)\right|\right|^2_{L^2(\Omega\times\Gamma)}\\ &\qquad\le\left|\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(0)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(0)\right)\right|\right|^2_{L^2(\Omega\times\Gamma)}\\ &\qquad\quad+K_1\int_0^t\left|\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(\theta)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(\theta)\right)\right|\right|^2_{L^2(\Omega\times\Gamma)} d\theta.\end{aligned} \tag{4.2.232}$$

The application of Gronwall's inequality yields[42]

$$\begin{aligned}\int_{\Omega\times\Gamma}&\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(t,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(t,x,y)\right)\right|^2 dx\,dy\\ &\le\left(1+K_1te^{K_1t}\right)\int_{\Omega\times\Gamma}\left|T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(0,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(0,x,y)\right)\right|^2 dx\,dy\ \ \forall\, t\end{aligned}$$

[41] We have used the *Lipschitz continuity* of ψ_δ.

[42] Confer part (v) of the theorem 3.6.4.

$$
\begin{aligned}
&\int_0^T \int_{\Omega\times\Gamma} \left| T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(t,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(t,x,y)\right)\right|^2 dx\,dy\,dt \\
&\quad \le \int_{\Omega\times\Gamma} \left| T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(0,x,y)\right) - T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(0,x,y)\right)\right|^2 dx\,dy \underbrace{\int_0^T \left(1+K_1 t e^{K_1 t}\right) dt}_{\text{a bounded quantity}=:C_{61}} \\
&\quad \le C_{61}\left[\int_{\Omega\times\Gamma} \left| T_b^{\varepsilon_\mu}\left(w_{\varepsilon_{\mu_{\delta_m}}}(0,x,y)\right)\right|^2 dx\,dy + \int_{\Omega\times\Gamma} \left| T_b^{\varepsilon_\nu}\left(w_{\varepsilon_{\nu_{\delta_m}}}(0,x,y)\right)\right|^2 dx\,dy\right] \\
&\quad \le C_{61}\left[\varepsilon_\mu \int_{\Gamma_{\varepsilon_\mu}} \left|w_{\varepsilon_{\mu_{\delta_m}}}(0)\right|^2 d\sigma_x + \varepsilon_\nu \int_{\Gamma_{\varepsilon_\nu}} \left|w_{\varepsilon_{\nu_{\delta_m}}}(0)\right|^2 d\sigma_x\right] \\
&\quad \le C_{61}\left[\varepsilon_\mu \int_{\Gamma_{\varepsilon_\mu}} \left\|w_{\varepsilon_{\mu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\mu})} d\sigma_x + \varepsilon_\nu \int_{\Gamma_{\varepsilon_\nu}} \left\|w_{\varepsilon_{\nu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\nu})} d\sigma_x\right] \\
&\quad \le C_{61}\left[\varepsilon_\mu \left\|w_{\varepsilon_{\mu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\mu})} \int_{\cup_{k=1}^{N_\varepsilon}\varepsilon_\mu\Gamma_k} d\sigma_x + \varepsilon_\nu \left\|w_{\varepsilon_{\nu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\nu})} \int_{\cup_{k=1}^{N_\varepsilon}\varepsilon_\nu\Gamma_k} d\sigma_x\right] \\
&\quad \le C_{61}\sum_{k=1}^{N_\varepsilon}\left[\varepsilon_\mu \left\|w_{\varepsilon_{\mu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\mu})} \int_{\varepsilon_\mu\Gamma_k} d\sigma_x + \varepsilon_\nu \left\|w_{\varepsilon_{\nu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\nu})} \int_{\varepsilon_\nu\Gamma_k} d\sigma_x\right] \\
&\quad \le C_{61}\sum_{k=1}^{N_\varepsilon}\left[\varepsilon_\mu \left\|w_{\varepsilon_{\mu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\mu})} \int_{\Gamma_k} \varepsilon_\mu^{n-1}\, d\sigma_y + \varepsilon_\nu \left\|w_{\varepsilon_{\nu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\nu})} \int_{\Gamma_k} \varepsilon_\nu^{n-1}\, d\sigma_y\right] \\
&\quad \le C_{61}\left(\sum_{k=1}^{N_\varepsilon}\int_{\Gamma_k} d\sigma_y\right)\max(\left\|w_{\varepsilon_{\mu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\mu})}, \left\|w_{\varepsilon_{\nu_{\delta_m}}}(0)\right\|^2_{L^\infty(\Gamma_{\varepsilon_\nu})})(\varepsilon_\mu^n+\varepsilon_\nu^n) \\
&\quad \le C_{61}\left(\varepsilon_\mu^n+\varepsilon_\nu^n\right) \to 0 \text{ as } \mu,\nu\to\infty.
\end{aligned} \tag{4.2.233}
$$

This shows that $\left(T_b^\varepsilon w_{\varepsilon_{\delta_m}}\right)_{\varepsilon>0}$ is a Cauchy sequence in $L^2((0,T)\times\Omega\times\Gamma)$. It is strongly convergent to a limit ξ in $L^2((0,T)\times\Omega\times\Gamma)$ ♦

The above lemma shows that $\left(T_b^\varepsilon w_{\varepsilon_{\delta_m}}\right)_{\varepsilon>0}$ is weakly convergent to ξ in $L^2((0,T)\times\Omega\times\Gamma)$. Since the weak limit of an unfolded sequence is equal to the two-scale limit of the sequence, $\xi = w_{\delta_m}$ (cf. part (vii) of the theorem 3.6.4). Furthermore the continuity of $\psi_\delta(T_b^\varepsilon w_{\varepsilon_{\delta_m}})$ implies its strong convergence to $\psi_\delta(w_{\delta_m})$ in $L^2((0,T)\times\Omega\times\Gamma)$. Under similar arguments $\psi_\delta(w_{\varepsilon_{\delta_m}})$ is two-scale convergent to $\psi_\delta(w_{\delta_m})$ in $L^2((0,T)\times\Omega\times\Gamma)$.

Therefore for all $m=1,2,\dots,I_2$, the sequences $(w_{\varepsilon_{\delta_m}})_{\varepsilon>0}$ and $(\psi_\delta(w_{\varepsilon_{\delta_m}}))_{\varepsilon>0}$ are *two-scale* convergent to the limits w_{δ_m} and $\psi_\delta(w_{\delta_m})$ respectively. (4.2.234)

Let us choose a test function $\phi(t,x,\frac{x}{\varepsilon}) \in [C_0^\infty((0,T)\times\Omega; C_{per}^\infty(Y))]^{I_2}$, then from (4.2.234) and from theorems 3.5.16 and 4.2.2.1.1.1, we have

$$
\begin{aligned}
&\sum_{m=1}^{I_2}\lim_{\varepsilon\to0}\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\frac{\partial w_{\varepsilon_{\delta_m}}(t,x)}{\partial t}\phi_m(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt \\
&\qquad = -k_d\sum_{m=1}^{I_2}\lim_{\varepsilon\to0}\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\psi_\delta(w_{\varepsilon_{\delta_m}}(t,x))\phi_m(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt,
\end{aligned}
$$

i.e.,

$$
\sum_{m=1}^{I_2}\int_0^T\int_{\Omega\times\Gamma}\frac{\partial w_{\delta_m}}{\partial t}\phi_m(t,x,y)\,dx\,dy\,dt = -k_d\sum_{m=1}^{I_2}\int_0^T\int_{\Omega\times\Gamma}\psi_\delta(w_{\delta_m}(t,x,y))\phi_m(t,x,y)\,dx\,dy\,dt,
$$

i.e.,

$$\int_0^T \int_{\Omega\times\Gamma} \left\langle \frac{\partial w_\delta(t,x,y)}{\partial t}, \phi(t,x,y) \right\rangle dx\, dy\, dt = -k_d \int_0^T \int_{\Omega\times\Gamma} \langle \psi_\delta(w_\delta(t,x,y)), \phi(t,x,y)\rangle\, dx\, dy\, dt,$$

$$\implies \quad \frac{\partial w_\delta}{\partial t} = -k_d \psi_\delta(w_\delta) \quad \text{for a.e. in} \quad (0,T)\times\Omega\times\Gamma, \tag{4.2.235}$$

$$w_\delta(0,x,y) = w_0(x,y) \quad \text{for a.e. in} \quad \Omega\times\Gamma.^{43} \tag{4.2.236}$$

4.2.2.3.2 Homogenization of the PDE (4.2.17)-(4.2.21)

Let us choose the functions $\phi_0 \in C_0^\infty((0,T)\times\Omega)^{I_2}$ and $\phi_1 \in C_0^\infty((0,T)\times\Omega; C_{per}^\infty(Y))^{I_2}$. Set $\phi(t,x,\frac{x}{\varepsilon}) = \phi_0(t,x) + \varepsilon\phi_1(t,x,\frac{x}{\varepsilon}) \in C_0^\infty((0,T)\times\Omega; C_{per}^\infty(Y))^{I_2}$. Using ϕ as test function in the weak formulation of (4.2.17)-(4.2.21), we get

$$\sum_{k=1}^{I_2} \int_0^T \left\langle \frac{\partial v_{\varepsilon\delta_k}}{\partial t}, \phi_k \right\rangle dt + \sum_{k=1}^{I_2} \int_0^T \int_{\Omega_\varepsilon^p} \left(D\nabla v_{\varepsilon\delta_k} - \vec{q}_\varepsilon v_{\varepsilon\delta_k} \right) \nabla\phi_k\, dx\, dt$$
$$+ \sum_{k=1}^{I_2} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon\delta_k}}{\partial t} \phi_k\, d\sigma_x\, dt = \sum_{k=1}^{I_2} \int_0^T \langle S_2 R(u_{\varepsilon\delta}, v_{\varepsilon\delta})_k, \phi_k\rangle\, dt,$$

i.e.,

$$I_{time} + I_{diff} + I_{bound} = I_{reac}, \tag{4.2.237}$$

where

$$I_{time} = \sum_{k=1}^{I_2} \int_0^T \left\langle \frac{\partial v_{\varepsilon\delta_k}}{\partial t}, \phi_k \right\rangle dt, \tag{4.2.238}$$

$$I_{diff} = \sum_{k=1}^{I_2} \int_0^T \int_{\Omega_\varepsilon^p} \left(D\nabla v_{\varepsilon\delta_k} - \vec{q}_\varepsilon v_{\varepsilon\delta_k} \right) \nabla\phi_k\, dx\, dt, \tag{4.2.239}$$

$$I_{bound} = \sum_{k=1}^{I_2} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon\delta_k}}{\partial t} \phi_k\, d\sigma_x\, dt, \tag{4.2.240}$$

$$I_{reac} = \sum_{k=1}^{I_2} \int_0^T \langle S_2 R(u_{\varepsilon\delta}, v_{\varepsilon\delta})_k, \phi_k\rangle\, dt. \tag{4.2.241}$$

Now we pass to the *two-scale* limit in each term separately. Note that for (4.2.238) and (4.2.241) the procedure to obtain the *two-scale* limit follows like the section 4.1.2.3 and we finally arrive to the equations similar to (4.1.100) and (4.1.102). Thus we have

$$\lim_{\varepsilon\to 0} I_{time} = \lim_{\varepsilon\to 0} \sum_{k=1}^{I_2} \int_0^T \left\langle \frac{\partial v_{\varepsilon\delta_k}}{\partial t}, \phi_k \right\rangle dt = |Y^p| \sum_{k=1}^{I_2} \int_0^T \left\langle \frac{\partial v_{\delta_k}}{\partial t}, \phi_{0_k} \right\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)} dt \tag{4.2.242}$$

[43]

$$||w_{0_m}||^p_{L^p(\Omega\times\Gamma)} = \int_\Omega \int_\Gamma \left| T_b^\varepsilon w_{0_m}(x,y) \right|^p dx\, d\sigma_y = |\Gamma| \int_\Omega |w_{0_m}(x)|^p\, dx < \infty \text{ by } (4.2.5).$$

and

$$\lim_{\varepsilon\to 0} I_{reac} = |Y^p|\int_0^T \langle S_2R(u_\delta,v_\delta),\phi_0\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt. \tag{4.2.243}$$

Next,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} I_{diff}\\
&= \lim_{\varepsilon\to 0}\sum_{k=1}^{I_2}\int_0^T\int_{\Omega_\varepsilon^p}\left(D\nabla v_{\varepsilon_{\delta_k}}-\vec{q}_\varepsilon v_{\varepsilon_{\delta_k}}\right)\left(\nabla\phi_{0_k}+\nabla_y\phi_{1_k}+\varepsilon\nabla\phi_{1_k}\right)dx\,dt\\
&= \lim_{\varepsilon\to 0}\sum_{k=1}^{I_2}\int_0^T\int_\Omega\chi\left(\frac{x}{\varepsilon}\right)\left(D\nabla v_{\varepsilon_{\delta_k}}-\vec{q}_\varepsilon v_{\varepsilon_{\delta_k}}\right)\left(\nabla\phi_{0_k}+\nabla_y\phi_{1_k}\right)dx\,dt\\
&\qquad+\underbrace{\lim_{\varepsilon\to 0}\varepsilon\sum_{k=1}^{I_2}\int_0^T\int_\Omega\chi\left(\frac{x}{\varepsilon}\right)\left(D\nabla v_{\varepsilon_{\delta_k}}-\vec{q}_\varepsilon v_{\varepsilon_{\delta_k}}\right)\nabla\phi_{1_k}\,dx\,dt}_{\to 0 \text{ as } \varepsilon\to 0}\\
&= \sum_{k=1}^{I_2}\int_0^T\int_\Omega\int_Y\chi(y)\left(D\left(\nabla v_{\delta_k}+\nabla_y v^1_{\delta_k}\right)-v_{\delta_k}\vec{q}_1\right)\left(\nabla\phi_{0_k}+\nabla_y\phi_{1_k}\right)dx\,dy\,dt\\
&= \sum_{k=1}^{I_2}\int_0^T\int_\Omega\int_{Y^p}\left(D\left(\nabla v_{\delta_k}+\nabla_y v^1_{\delta_k}\right)-v_{\delta_k}\vec{q}_1\right)\left(\nabla\phi_{0_k}+\nabla_y\phi_{1_k}\right)dx\,dy\,dt.
\end{aligned}\tag{4.2.244}$$

Again,

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_{bound} &= \lim_{\varepsilon\to 0}\sum_{k=1}^{I_2}\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\frac{\partial w_{\varepsilon_{\delta_k}}}{\partial t}\phi_k\,d\sigma_x\,dt\\
&= \lim_{\varepsilon\to 0}\sum_{k=1}^{I_2}\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\frac{\partial w_{\varepsilon_{\delta_k}}}{\partial t}\phi_{0_k}\,d\sigma_x\,dt+\lim_{\varepsilon\to 0}\varepsilon\underbrace{\sum_{k=1}^{I_2}\varepsilon\int_0^T\int_{\Gamma_\varepsilon}\frac{\partial w_{\varepsilon_{\delta_k}}}{\partial t}\phi_{1_k}\,d\sigma_x\,dt}_{\to 0 \text{ as } \varepsilon\to 0}\\
&= \sum_{k=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma\frac{\partial w_{\delta_k}}{\partial t}\phi_{0_k}\,dx\,dy\,dt\\
&= \int_0^T\int_\Omega\int_\Gamma\left\langle\frac{\partial w_\delta}{\partial t},\phi_0\right\rangle dx\,dy\,dt.
\end{aligned}\tag{4.2.245}$$

Combining the equations (4.2.242), (4.2.243), (4.2.244) and (4.2.245), we obtain

$$\begin{aligned}
&|Y^p|\int_0^T\left\langle\frac{\partial v_\delta}{\partial t},\phi_0\right\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}dt\\
&+\sum_{k=1}^{I_2}\int_0^T\int_\Omega\int_{Y^p}\left(D\left(\nabla v_{\delta_k}+\nabla_y v^1_{\delta_k}\right)-v_{\delta_k}\vec{q}_1\right)\left(\nabla\phi_{0_k}+\nabla_y\phi_{1_k}\right)dx\,dy\,dt\\
&\quad+\int_0^T\int_\Omega\int_\Gamma\left\langle\frac{\partial w_\delta}{\partial t},\phi_0\right\rangle dx\,dy\,dt=|Y^p|\int_0^T\langle S_2R(u_\delta,v_\delta),\phi_0\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt.
\end{aligned}\tag{4.2.246}$$

In an analogy to section 4.1.2.3, here also we decouple the equation (4.2.246) to achieve the homogenized equation and the *Cell-Problem*. Setting $\phi_0\equiv 0$, the equation (4.2.246) reduces to

$$\sum_{k=1}^{I_2}\int_0^T\int_\Omega\int_{Y^p}\left(D\left(\nabla v_{\delta_k}+\nabla_y v^1_{\delta_k}\right)-v_{\delta_k}\vec{q}_1\right)\cdot\nabla_y\phi_{1_k}\,dx\,dy\,dt=0, \tag{4.2.247}$$

We state the following lemma from [MZ11]:

Lemma 4.2.2.3.2.1. *Let $a_j(y)$ for $j=1,2,...,n$ be the $Y-$ periodic solution of the integral identity*

$$\int_0^T \int_\Omega \int_{Y^p} (e_j + \nabla_y a_j(y)) \cdot \nabla_y \phi_{1_k} \, dx\, dy\, dt = 0, \tag{4.2.248}$$

and $a_0(t,x,y)$ be the solution to the integral identity

$$\int_0^T \int_\Omega \int_{Y^p} (\vec{q}_1 + \nabla_y a_0) \cdot \nabla_y \phi_{1_k} \, dx\, dy\, dt = 0, \tag{4.2.249}$$

for any $Y-$ periodic smooth function ϕ_1. Then the function

$$v^1_{\delta_k}(x,y,t) = \sum_{j=1}^n \frac{\partial v_{\delta_k}(t,x)}{\partial x_j} a_j(y) + a_0(x,y,t) v_{\delta_k}(t,x)$$

satisfies the integral identity (4.2.247).

Proof. The proof is straightforward and hence omitted. ♦

We set $\vec{q}_0 = D\int_{Y^p} \nabla_y a_0 \, dy$. Now setting $\phi_1 \equiv 0$, then (4.2.246) reduces to

$$|Y^p| \int_0^T \left\langle \frac{\partial v_\delta}{\partial t}, \phi_0 \right\rangle dt + \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} \left(D\left(\nabla v_{\delta_k} + \nabla_y v^1_{\delta_k} \right) - v_{\delta_k} \vec{q}_1 \right) \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$+ \int_0^T \int_\Omega \int_\Gamma \left\langle \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx\, dy\, dt = |Y^p| \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle \, dt,$$

i.e.,

$$\int_0^T \left\langle \frac{\partial v_\delta}{\partial t}, \phi_0 \right\rangle dt + \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} D\left(\nabla v_{\delta_k} + \nabla_y v^1_{\delta_k} \right) \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$- \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} v_{\delta_k} \vec{q}_1 \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx\, dy\, dt,$$

i.e.,

$$\int_0^T \left\langle \frac{\partial v_\delta}{\partial t}, \phi_0 \right\rangle dt + \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} \frac{D}{|Y^p|} \left(\nabla v_{\delta_k} + \nabla_y v^1_{\delta_k} \right) \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$- \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} v_{\delta_k} \vec{q}_1 \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx\, dy\, dt. \tag{4.2.250}$$

Substituting $v^1_{\delta_k}(x,y,t) = \sum_{j=1}^n \frac{\partial v_{\delta_k}(t,x)}{\partial x_j} a_j(y) + a_0(x,y,t) v_{\delta_k}(t,x)$, for $k=1,2,...,I_2$, in (4.2.250) leaves

$$\int_0^T \left\langle \frac{\partial v_\delta}{\partial t}, \phi_0 \right\rangle dt + \int_0^T \int_\Omega \int_{Y^p} \frac{D}{|Y^p|} \left(\nabla v_{\delta_k} + \sum_{j=1}^n \frac{\partial v_{\delta_k}}{\partial x_j} \nabla_y a_j + \nabla_y a_0 v_{\delta_k} \right) \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$- \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} v_{\delta_k} \vec{q}_1 \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx\, dy\, dt,$$

i.e.,

$$\int_0^T \left\langle \frac{\partial v_\delta}{\partial t}, \phi_0 \right\rangle dt + \sum_{k=1}^{I_2} \int_0^T \int_\Omega \sum_{l,j=1}^{n} \left(\frac{D}{|Y^p|} \int_{Y^p} \left(\delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \right) \frac{\partial v_{\delta_k}}{\partial x_j} \frac{\partial \phi_{0_k}}{\partial x_l} \, dx\, dt$$
$$- \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_\Omega \int_{Y^p} (\vec{q}_1 - D\nabla_y a_0) v_{\delta_k} \nabla \phi_{0_k} \, dx\, dy\, dt$$
$$= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx\, dy\, dt,$$

i.e.,

$$\int_0^T \left\langle \frac{\partial v_\delta}{\partial t}, \phi_0 \right\rangle dt + \sum_{k=1}^{I_2} \int_0^T \int_\Omega P \nabla v_{\delta_k} \cdot \nabla \phi_{0_k} \, dx\, dt - \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_\Omega (\vec{q} - \vec{q}_0) v_{\delta_k} \nabla \phi_{0_k} \, dx\, dt$$
$$= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx\, dy\, dt, \tag{4.2.251}$$

where $P = (p_{jl})_{\substack{1 \le j \le n \\ 1 \le l \le n}}$ is a positive definite second order symmetric tensor whose components are given by

$$p_{jl} = \int_{Y^p} \frac{D}{|Y^p|} \left(\delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \quad \text{for } j, l = 1, 2, \ldots, n. \tag{4.2.252}$$

Therefore the strong form of the homogenized equation (4.2.251) is

$$\boxed{\begin{aligned}
&\frac{\partial v_\delta}{\partial t} - \nabla \left(P \nabla v_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) v_\delta \right) = S_2 R(u_\delta, v_\delta) \\
&\qquad\qquad - \frac{1}{|Y^p|} \int_\Gamma \frac{\partial w_\delta}{\partial t} dy \quad \text{in} \quad (0,T) \times \Omega, && (4.2.253) \\
&- \left(P \nabla v_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) v_\delta \right) \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_{in}, && (4.2.254) \\
&- \left(P \nabla v_\delta + \frac{1}{|Y^p|} \vec{q}_0 v_\delta \right) \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_{out}, && (4.2.255) \\
&v_\delta(0,x) = v_0(x) \quad \text{in} \quad \Omega. && (4.2.256)
\end{aligned}}$$

4.2.2.3.3 Homogenization of the PDE (4.2.12)-(4.2.16)

The arguments and the procedure to homogenize the PDE (4.2.12)-(4.2.16) are similar to the approach shown in the previous subsection 4.2.2.3.2. Choosing a function $\phi(t,x,\frac{x}{\varepsilon}) = \phi_0(t,x) + \varepsilon \phi_1(t,x,\frac{x}{\varepsilon})$, where $\phi_0 \in C_0^\infty((0,T) \times \Omega)^{I_1}$ and $\phi_1 \in C_0^\infty((0,T) \times \Omega; C_{per}^\infty(Y))^{I_1}$. Using ϕ as test function in the weak formulation of the PDE (4.2.12)-(4.2.16), we get

$$\sum_{i=1}^{I_1} \int_0^T \left\langle \frac{\partial u_{\varepsilon_{\delta_i}}}{\partial t}, \phi_i \right\rangle dt + \sum_{i=1}^{I_1} \int_0^T \int_{\Omega_\varepsilon^p} \left(D \nabla u_{\varepsilon_{\delta_i}} - \vec{q}_\varepsilon v_{\varepsilon_{\delta_i}} \right) \nabla \phi_i \, dx\, dt = \sum_{i=1}^{I_1} \int_0^T \langle S_1 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta})_i, \phi_i \rangle \, dt,$$

i.e.,

$$I_{time} + I_{diff} = I_{reac}, \tag{4.2.257}$$

where

$$I_{time} = \sum_{i=1}^{I_1} \int_0^T \left\langle \frac{\partial v_{\varepsilon_{\delta_i}}}{\partial t}, \phi_i \right\rangle dt, \tag{4.2.258}$$

$$I_{diff} = \sum_{i=1}^{I_1} \int_0^T \int_{\Omega_\varepsilon^p} \left(D\nabla v_{\varepsilon_{\delta_i}} - \vec{q}_\varepsilon v_{\varepsilon_{\delta_i}} \right) \nabla \phi_i \, dx \, dt, \tag{4.2.259}$$

$$I_{reac} = \sum_{i=1}^{I_1} \int_0^T \langle S_1 R(u_{\varepsilon_\delta}, v_{\varepsilon_\delta})_i, \phi_i \rangle \, dt. \tag{4.2.260}$$

Letting $\varepsilon \to 0$ in *two-scale* sense in the terms I_{time}, I_{diff} and I_{reac} and proceeding in a similar fashion like the previous subsection, we obtain weak form of the homogenized equation as

$$\int_0^T \left\langle \frac{\partial u_\delta}{\partial t}, \phi_0 \right\rangle dt + \sum_{i=1}^{I_1} \int_0^T \int_\Omega P\nabla u_{\delta_i} \cdot \nabla \phi_{0_i} \, dx \, dt$$
$$- \frac{1}{|Y^p|} \sum_{i=1}^{I_1} \int_0^T \int_\Omega (\vec{q} - \vec{q}_0) u_{\delta_i} \nabla \phi_{0_i} \, dx \, dt = \int_0^T \langle S_1 R(u_\delta, v_\delta), \phi_0 \rangle \, dt, \tag{4.2.261}$$

where $P = (p_{jl})_{\substack{1\le j\le n \\ 1\le l\le n}}$ is a positive definite second order symmetric tensor whose components are given by

$$p_{jl} = \int_{Y^p} \frac{D}{|Y^p|} \left(\delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \text{ for } j,l = 1,2,\ldots,n. \tag{4.2.262}$$

The strong form of the homogenized problem (4.2.261) is

$$\boxed{\begin{aligned}
\frac{\partial u_\delta}{\partial t} - \nabla \left(P\nabla u_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) u_\delta \right) &= S_1 R(u_\delta, v_\delta) && \text{in} \quad (0,T) \times \Omega, && (4.2.263)\\
-\left(P\nabla u_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) u_\delta \right) \cdot \vec{n} &= d && \text{on} \quad (0,T) \times \partial\Omega_{in}, && (4.2.264)\\
-\left(P\nabla u_\delta + \frac{1}{|Y^p|} \vec{q}_0 u_\delta \right) \cdot \vec{n} &= 0 && \text{on} \quad (0,T) \times \partial\Omega_{out}, && (4.2.265)\\
u_\delta(0,x) &= u_0(x) && \text{in} \quad \Omega. && (4.2.266)
\end{aligned}}$$

Therefore the complete homogenized problem is

$$\frac{\partial u_\delta}{\partial t} - \nabla \left(P\nabla u_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) u_\delta \right) = S_1 R(u_\delta, v_\delta) \quad \text{in} \quad (0,T) \times \Omega, \tag{4.2.267}$$

$$-\left(P\nabla u_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) u_\delta \right) \cdot \vec{n} = d \quad \text{on} \quad (0,T) \times \partial\Omega_{in}, \tag{4.2.268}$$

$$-\left(P\nabla u_\delta + \frac{1}{|Y^p|} \vec{q}_0 u_\delta \right) \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_{out}, \tag{4.2.269}$$

$$u_\delta(0,x) = u_0(x) \quad \text{in} \quad \Omega. \tag{4.2.270}$$

$$\frac{\partial v_\delta}{\partial t} - \nabla \left(P\nabla v_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) v_\delta \right) = S_2 R(u_\delta, v_\delta)$$
$$- \frac{1}{|Y^p|} \int_\Gamma \frac{\partial w_\delta}{\partial t} dy \text{ in} \quad (0,T) \times \Omega, \tag{4.2.271}$$

$$-\left(P\nabla v_\delta - \frac{1}{|Y^p|} (\vec{q} - \vec{q}_0) v_\delta \right) \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_{in}, \tag{4.2.272}$$

$$-\left(P\nabla v_\delta + \frac{1}{|Y^p|} \vec{q}_0 v_\delta \right) \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_{out}, \tag{4.2.273}$$

$$v_\delta(0,x) = v_0(x) \quad \text{in} \quad \Omega, \tag{4.2.274}$$

$$\frac{\partial w_\delta}{\partial t} = -k_d \psi_\delta(w_\delta) \quad \text{on} \quad (0,T) \times \Omega \times \Gamma, \tag{4.2.275}$$

$$w_\delta(0,x,y) = w_0(x,y) \quad \text{on} \quad \Omega \times \Gamma, \tag{4.2.276}$$

where the solution

$$u_\delta \in \mathcal{F}_2^u \cap L^\infty((0,T);L^\infty(\Omega))^{I_1}, \quad v_\delta \in \mathcal{G}_2^v \cap L^\infty((0,T);L^\infty(\Omega))^{I_2} \quad \text{and } w_\delta \in \mathcal{H}_2^w. \tag{4.2.277}$$

The velocity vector satisfies

$$\nabla_y \cdot \vec{q}_1 = 0 \quad \text{in } (0,T)\times\Omega\times Y^p \text{ and } \nabla\cdot\int_{Y^p}\vec{q}_1\,dy = 0 \quad \text{in } (0,T)\times\Omega, \tag{4.2.278}$$

$$\vec{q}_1 = 0 \quad \text{in } (0,T)\times\Omega\times Y^s. \tag{4.2.279}$$

4.2.2.3.4 Uniqueness of the Solution of (4.2.267)-(4.2.276)

Theorem 4.2.2.3.4.1. *There exists a unique solution of the homogenized problem (4.2.267)-(4.2.276).*

Proof. Following the steps of theorem 4.2.1.5.1 yields the proof. Note that P is a second order positive definite symmetric tensor. ♦

4.2.3 Passage to the Limit as $\delta \to 0$ in the Problem (P_δ^2)

Theorem 4.2.3.1. *For any $\delta > 0$, the solution $(u_\delta, v_\delta, w_\delta)$ of the problem (4.2.267)-(4.2.276) satisfies the following estimate:*

$$\begin{aligned}
&|||u_\delta|||_{L^2((0,T);L^2(\Omega))^{I_1}} + |||u_\delta|||_{L^\infty((0,T);L^\infty(\Omega))^{I_1}} + |||\nabla u_\delta|||_{L^2((0,T);L^2(\Omega))^{I_1}}\\
&+\left|\left|\left|\frac{\partial u_\delta}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}} + |||v_\delta|||_{L^2((0,T);L^2(\Omega))^{I_2}} + |||v_\delta|||_{L^\infty((0,T);L^\infty(\Omega))^{I_2}}\\
&+|||\nabla v_\delta|||_{L^2((0,T);L^2(\Omega))^{I_2}} + \left|\left|\left|\frac{\partial v_\delta}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}} + |||w_\delta|||_{L^p((0,T)\times\Omega\times\Gamma)^{I_2}}\\
&+|||w_\delta|||_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} + \left|\left|\left|\frac{\partial w_\delta}{\partial t}\right|\right|\right|_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} + \left|\left|\left|\int_\Gamma \frac{\partial w_\delta(y)}{\partial t}\,d\sigma_y\right|\right|\right|_{L^2((0,T)\times\Omega)^{I_2}}\\
&\le C_{62} < \infty,
\end{aligned}\tag{4.2.280}$$

where C_{62} is independent of δ.

Proof. The proof consists of several steps.

(i) Multiplying both sides of (4.2.275) by $\frac{\partial w_\delta}{\partial t}$ and integrating, we obtain

$$\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma\left|\frac{\partial w_{\delta_m}}{\partial t}\right|^2 dx\,d\sigma_y\,dt = -k_d\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma \psi_\delta(w_{\delta_m})\,\frac{\partial w_{\delta_m}}{\partial t}\,dx\,d\sigma_y\,dt,$$

i.e.,

$$\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma\left|\frac{\partial w_{\delta_m}}{\partial t}\right|^2 dx\,d\sigma_y\,dt \le \frac{1}{2}\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma\left[|k_d\,\psi_\delta(w_{\delta_m})|^2 + \left|\frac{\partial w_{\delta_m}}{\partial t}\right|^2\right] dx\,d\sigma_y\,dt,$$

i.e.,

$$\frac{1}{2}\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma\left|\frac{\partial w_{\delta_m}}{\partial t}\right|^2 dx\,d\sigma_y\,dt \le \frac{k_d^2}{2}\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma dx\,d\sigma_y\,dt \quad \text{since } |\psi_\delta(w_{\delta_m})| \le 1,$$

i.e.,

$$\left|\left|\left|\frac{\partial w_\delta}{\partial t}\right|\right|\right|^2_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} \le k_d^2\, T\, |\Omega|\, |\Gamma|\, I_2,$$

i.e.,

$$\left|\left|\left|\frac{\partial w_\delta}{\partial t}\right|\right|\right|_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} \le C_{63}, \tag{4.2.281}$$

where C_{63} $(:= (k_d^2\, T\, |\Omega|\, |\Gamma|\, I_2)^{\frac{1}{2}})$ is independent of δ.
(ii) Multiplying both sides of (4.2.275) by w_δ and integrating, we get

$$\sum_{m=1}^{I_2}\int_0^t\int_\Omega\int_\Gamma \frac{\partial w_{\delta_m}}{\partial\theta} w_{\delta_m}\, dx\, d\sigma_y\, d\theta = -k_d \sum_{m=1}^{I_2}\int_0^t\int_\Omega\int_\Gamma \psi_\delta(w_{\delta_m}) w_{\delta_m}\, dx\, d\sigma_y\, d\theta,$$

i.e.,

$$\frac{1}{2}\sum_{m=1}^{I_2}\int_0^t \frac{\partial}{\partial\theta}||w_{\delta_m}(\theta)||^2_{L^2(\Omega\times\Gamma)}\, d\theta \le \frac{1}{2}\sum_{m=1}^{I_2}\int_0^t\int_\Omega\int_\Gamma \left[|k_d\psi_\delta(w_{\delta_m})|^2 + |w_{\delta_m}|^2\right] dx\, d\sigma_y\, d\theta,$$

i.e.,

$$\sum_{m=1}^{I_2}\left[||w_{\delta_m}(t)||^2_{L^2(\Omega\times\Gamma)} - ||w_m(0)||^2_{L^2(\Omega\times\Gamma)}\right] \le \sum_{m=1}^{I_2}\int_0^t\int_\Omega\int_\Gamma \left[k_d^2 + |w_{\delta_m}|^2\right] dx\, d\sigma_y\, d\theta.$$

This gives

$$\sum_{m=1}^{I_2}||w_{\delta_m}(t)||^2_{L^2(\Omega\times\Gamma)} \le \sum_{m=1}^{I_2}||w_{0_m}||^2_{L^2(\Omega\times\Gamma)} + I_2\, k_d^2\, T\, |\Omega|\, |\Gamma| + \int_0^t \sum_{m=1}^{I_2}||w_{\delta_m}(\theta)||^2_{L^2(\Omega\times\Gamma)}\, d\theta,$$

i.e.,

$$\sum_{m=1}^{I_2}||w_{\delta_m}(t)||^2_{L^2(\Omega\times\Gamma)} \le |||w_0|||^2_{L^2(\Omega\times\Gamma)^{I_2}} + I_2\, k_d^2\, T\, |\Omega|\, |\Gamma| + \int_0^t \sum_{m=1}^{I_2}||w_{\delta_m}(\theta)||^2_{L^2(\Omega\times\Gamma)}\, d\theta,$$

i.e.,

$$\sum_{m=1}^{I_2}||w_{\delta_m}(t)||^2_{L^2(\Omega\times\Gamma)} \le C_{64} + \int_0^t \sum_{m=1}^{I_2}||w_{\delta_m}(\theta)||^2_{L^2(\Omega\times\Gamma)}\, d\theta,$$

where C_{64} $(:= |||w_0|||^2_{L^2(\Omega\times\Gamma)^{I_2}} + I_2\, k_d^2\, T\, |\Omega|\, |\Gamma|)$ is independent of δ by footnote 43. Gronwall's inequality yields

$$\sum_{m=1}^{I_2}||w_{\delta_m}(t)||^2_{L^2(\Omega\times\Gamma)} \le C_{64}(1 + te^t),$$

i.e.,

$$\sum_{m=1}^{I_2}\int_0^T ||w_{\delta_m}(t)||^2_{L^2(\Omega\times\Gamma)}\, dt \le \int_0^T C_{64}(1 + te^t)\, dt =: C_{65},$$

i.e.,

$$|||w_\delta|||_{L^2((0,T)\times\Omega\times\Gamma)} \le C_{65}, \tag{4.2.282}$$

where C_{65} is independent of δ.
(iii) Multiplying both sides of the *m-th* ODE of (4.2.275) by $w_{\delta_m}|w_{\delta_m}|^{p-2}$ and integrating

$$\int_0^t\int_\Omega\int_\Gamma w_{\delta_m}(\theta,x)\,|w_{\delta_m}(\theta,x)|^{p-2}\,\frac{\partial w_{\delta_m}(\theta,x)}{\partial\theta}\,dx\,d\sigma_y\,d\theta$$
$$= -k_d\int_0^t\int_\Omega\int_\Gamma w_{\delta_m}(\theta,x)\,|w_{\delta_m}(\theta,x)|^{p-2}\,\psi_\delta(w_{\delta_m}(\theta,x))\,dx\,d\sigma_y\,d\theta,$$

i.e.,

$$\int_0^t\int_\Omega\int_\Gamma\frac{1}{p}\frac{\partial}{\partial\theta}\,|w_{\delta_m}(\theta,x)|^p\,dx\,d\sigma_y\,d\theta\le\int_0^t\int_\Omega\int_\Gamma\left[\frac{p-1}{p}\,|w_{\delta_m}(\theta,x)|^p+\frac{k_d^p}{p}\right]dx\,d\sigma_y\,d\theta,$$

i.e.,

$$\int_\Omega\int_\Gamma|w_{\delta_m}(t,x)|^p\,dx\,d\sigma_y\le\int_\Omega\int_\Gamma|w_{\delta_m}(0,x)|^p\,dx\,d\sigma_y+\int_0^t\int_\Omega\int_\Gamma\left[(p-1)\,|w_{\delta_m}(\theta,x)|^p+k_d^p\right]dx\,d\sigma_y\,d\theta,$$

i.e.,

$$\sum_{m=1}^{I_2}\int_\Omega\int_\Gamma|w_{\delta_m}(t,x)|^p\,dx\,d\sigma_y$$
$$\le\sum_{m=1}^{I_2}\int_\Omega\int_\Gamma|w_{0_m}(x)|^p\,dx\,d\sigma_y+T\,I_2\,|\Gamma|\,|\Omega|\,k_d^p+(p-1)\sum_{m=1}^{I_2}\int_0^t\int_\Omega\int_\Gamma|w_{\delta_m}(\theta,x)|^p\,dx\,d\sigma_y\,d\theta.$$

A straightforward application of Gronwall's inequality and steps similar to part (ii) will imply

$$|||w_\delta|||_{L^p((0,T)\times\Omega\times\Gamma)^{I_2}}=\left[\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma|w_{\delta_m}(t,x)|^p\,dx\,d\sigma_y\,dt\right]^{\frac{1}{p}}\le C_{66},\tag{4.2.283}$$

where C_{66} is independent of δ.

(iv) Integrating both sides of *m-th* ODE of (4.2.275) and squaring leaves

$$\left[\int_\Gamma\frac{\partial w_{\delta_m}}{\partial t}\,d\sigma_y\right]^2=k_d^2\left[\int_\Gamma\psi_d(w_{\delta_m})\,d\sigma_y\right]^2,$$

i.e.,

$$\left|\int_\Gamma\frac{\partial w_{\delta_m}}{\partial t}\,d\sigma_y\right|^2\le k_d^2\,|\Gamma|^2,$$

i.e.,

$$\left\|\int_\Gamma\frac{\partial w_{\delta_m}}{\partial t}\,d\sigma_y\right\|^2_{L^2((0,T)\times\Omega)}\le|\Omega|\;|\Gamma|^2\;T\,k_d^2,$$

i.e.,

$$\sum_{m=1}^{I_2}\left\|\int_\Gamma\frac{\partial w_{\delta_m}}{\partial t}\,d\sigma_y\right\|^2_{L^2((0,T)\times\Omega)}\le I_2\;|\Omega|\;|\Gamma|^2\;T\,k_d^2,$$

i.e.,

$$\left|\left|\left|\int_\Gamma \frac{\partial w_\delta}{\partial t}\,d\sigma_y\right|\right|\right|_{L^2((0,T)\times\Omega)^{I_2}} \le C_{67}, \tag{4.2.284}$$

where C_{67} $(:= (I_2\ |\Omega|\ |\Gamma|^2\ T\ k_d^2)^{\frac{1}{2}})$ is independent of δ.
(v) From corollary 4.2.2.2.1.3, we have[44]

$$|||v_\delta|||_{L^\infty((0,T)\times\Omega)^{I_2}} \le C_{68}, \tag{4.2.285}$$

where C_{68} is independent of δ.
(vi) Note that

$$\begin{aligned}|||v_\delta|||^2_{L^2((0,T)\times\Omega)^{I_2}} = \sum_{k=1}^{I_2} ||v_{\delta_k}||^2_{L^2((0,T)\times\Omega)} &= \sum_{k=1}^{I_2}\int_0^T\int_\Omega |v_{\delta_k}(t,x)|^2\,dx\,dt\\ &\le \sum_{k=1}^{I_2} ||v_{\delta_k}||^2_{L^\infty((0,T)\times\Omega)}\ T\ |\Omega|\\ &\le |||v_\delta|||^2_{L^\infty((0,T)\times\Omega)^{I_2}}\ I_2\,T\ |\Omega|\\ &\le C_{68}^2\ I_2\,T\ |\Omega|, \quad \text{by (4.2.285)},\end{aligned}$$

i.e.,

$$|||v_\delta|||_{L^2((0,T)\times\Omega)^{I_2}} \le C_{69}, \tag{4.2.286}$$

where C_{69} $(:= (C_{68}^2\ I_2\,T\ |\Omega|)^{\frac{1}{2}})$ is independent of δ.
(vii) Using corollary 4.2.2.2.2.3, we get

$$|||u_\delta|||_{L^\infty((0,T)\times\Omega)^{I_1}} \le C_{70}, \tag{4.2.287}$$

where C_{70} is independent of δ.
(viii)

$$|||u_\delta|||_{L^2((0,T)\times\Omega)^{I_1}} \le C_{71}, \tag{4.2.288}$$

where C_{71} is independent of δ.
(ix) Testing (4.2.271) by v_δ leaves

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{I_2}\int_0^T \frac{d}{dt}||v_{\delta_k}(t)||^2_{L^2(\Omega)}\,dt + \sum_{k=1}^{I_2}\int_0^T\int_\Omega P\nabla v_{\delta_k}\cdot\nabla v_{\delta_k}\,dx\,dt\\ &= \sum_{k=1}^{I_2}\Bigg[\frac{1}{|Y^p|}\int_0^T\int_{\partial\Omega_{in}} \vec{q}\cdot\vec{n}\,|v_{\delta_k}|^2\,ds\,dt - \frac{1}{|Y^p|}\int_\Omega\int_0^T \vec{q}\cdot\nabla v_{\delta_k} v_{\delta_k}\,dx\,dt + \int_0^T \langle S_2R(u_\delta,v_\delta)_k, v_{\delta_k}\rangle\,dt\\ &\quad -\frac{1}{|Y^p|}\int_0^T\int_\Omega \vec{q}_0\cdot\nabla v_{\delta_k} v_{\delta_k}\,dx\,dt - \frac{1}{|Y^p|}\int_0^T\int_\Omega\left(\int_\Gamma \frac{\partial w_{\delta_k}}{\partial t}\,d\sigma_y\right) v_{\delta_k}\,dx\,dt\Bigg].\end{aligned} \tag{4.2.289}$$

Recall theorem 3.1.3 which implies $|||v(0)|||_{L^\infty(\Omega)^{I_2}} < \infty$ and investing the knowledge of

[44]Notice that v_δ is independent of y.

positive definiteness of the symmetric tensor P, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{I_2}||v_{\delta_k}(T)||^2_{L^2(\Omega)} - \frac{1}{2}\sum_{k=1}^{I_2}||v_k(0)||^2_{L^2(\Omega)} + \sum_{k=1}^{I_2}\int_0^T\int_\Omega \beta\,|\nabla v_{\delta_k}|^2\,dx\,dt \\
&\le \sum_{k=1}^{I_2}\left[\frac{||\vec{q}\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}}{|Y^p|}\int_0^T\int_{\partial\Omega}|v_{\delta_k}|^2\,ds\,dt + \frac{1}{|Y^p|}\int_0^T\int_\Omega|\vec{q}|\,|\nabla v_{\delta_k}|\,|v_{\delta_k}|\,dx\,dt\right] \\
&\quad + \frac{1}{2}\sum_{k=1}^{I_2}\left[\int_0^T||S_2R(u_\delta(t),v_\delta(t))_k||^2_{L^2(\Omega)}\,dt + \int_0^T||v_{\delta_k}(t)||^2_{L^2(\Omega)}\,dt\right. \\
&\left. + \frac{1}{|Y^p|}\int_0^T\int_\Omega|\vec{q}_0|\,|\nabla v_{\delta_k}|\,|v_{\delta_k}|\,dx\,dt\right] + \frac{1}{2}\sum_{k=1}^{I_2}\left[\int_0^T\int_\Omega\left|\int_\Gamma\frac{\partial w_{\delta_k}}{\partial t}\,d\sigma_y\right|^2dx\,dt + \int_0^T||v_{\delta_k}(t)||^2_{L^2(\Omega)}\,dt\right].
\end{aligned} \tag{4.2.290}$$

By Young's inequality,[45]

$$\begin{aligned}
&\frac{1}{|Y^p|}\sum_{k=1}^{I_2}\int_0^T\int_\Omega|\vec{q}|\,|\nabla v_{\delta_k}|\,|v_{\delta_k}|\,dx\,dt + \frac{1}{|Y^p|}\sum_{k=1}^{I_2}\int_0^T\int_\Omega|\vec{q}_0|\,|\nabla v_{\delta_k}|\,|v_{\delta_k}|\,dx\,dt \\
&\le \frac{1}{|Y^p|}\sum_{k=1}^{I_2}||\vec{q}||_{L^2((0,T)\times\Omega)}\,||\nabla v_{\delta_k}||_{L^2((0,T)\times\Omega)}\,||v_{\delta_k}||_{L^\infty((0,T)\times\Omega)} \\
&\qquad + \frac{1}{|Y^p|}\sum_{k=1}^{I_2}||\vec{q}_0||_{L^2((0,T)\times\Omega)}\,||\nabla v_{\delta_k}||_{L^2((0,T)\times\Omega)}\,||v_{\delta_k}||_{L^\infty((0,T)\times\Omega)} \\
&\le \frac{1}{|Y^p|}\sum_{k=1}^{I_2}\left(C_{46}^2 I_2\right)^{\frac{1}{2}}||\vec{q}||_{L^2((0,T)\times\Omega)}\,||\nabla v_{\delta_k}||_{L^2((0,T)\times\Omega)} \\
&\qquad + \frac{1}{|Y^p|}\sum_{k=1}^{I_2}\left(C_{46}^2 I_2\right)^{\frac{1}{2}}||\vec{q}||_{L^2((0,T)\times\Omega)}\,||\nabla v_{\delta_k}||_{L^2((0,T)\times\Omega)} \quad \text{by corollary 4.2.2.2.1.3} \\
&\le \sum_{k=1}^{I_2}\left[\frac{\beta}{4}||\nabla v_{\delta_k}||^2_{L^2((0,T)\times\Omega)} + \frac{(C_{46}^2 I_2)^{\frac{1}{2}}}{\beta\,|Y^p|}||\vec{q}||^2_{L^2((0,T)\times\Omega)}\right] \\
&\qquad + \sum_{k=1}^{I_2}\left[\frac{\beta}{4}||\nabla v_{\delta_k}||^2_{L^2((0,T)\times\Omega)} + \frac{(C_{46}^2 I_2)^{\frac{1}{2}}}{\beta\,|Y^p|}||\vec{q}_0||^2_{L^2((0,T)\times\Omega)}\right] \\
&\le \sum_{k=1}^{I_2}\left[\frac{\beta}{2}||\nabla v_{\delta_k}||^2_{L^2((0,T)\times\Omega)} + \frac{(C_{46}^2 I_2)^{\frac{1}{2}}}{\beta\,|Y^p|}\left(||\vec{q}||^2_{L^2((0,T)\times\Omega)} + ||\vec{q}_0||^2_{L^2((0,T)\times\Omega)}\right)\right].
\end{aligned} \tag{4.2.291}$$

From boundary inequality (cf. theorem B.7) and Young's inequality, we have

$$\begin{aligned}
&\frac{1}{|Y^p|}\sum_{k=1}^{I_2}||\vec{q}\cdot\vec{n}||_{L^\infty((0,T)\times\partial\Omega_{in})}\int_0^T\int_{\partial\Omega}|v_{\delta_k}|^2\,ds\,dt \\
&\le \frac{C_{72}\,||\vec{q}\cdot\vec{n}||_{L^\infty(\partial\Omega_{in})}}{|Y^p|}\sum_{k=1}^{I_2}\left[||\nabla v_{\delta_k}||_{L^2((0,T);L^2(\Omega))}\,||v_{\delta_k}||_{L^2((0,T);L^2(\Omega))} + ||v_{\delta_k}||^2_{L^2((0,T);L^2(\Omega))}\right]
\end{aligned}$$

[45] We note that $||\vec{q}||_{L^2((0,T)\times\Omega)} < \infty$ and $||\vec{q}_0||_{L^2((0,T)\times\Omega)} < \infty$.

$$\leq \frac{\beta}{4}\sum_{k=1}^{I_2}\int_0^T\int_\Omega |\nabla v_{\delta_k}|^2\,dx\,dt$$
$$+\underbrace{\left(\frac{(C_{72}\;||\vec{q}\cdot\vec{n}||_{L^\infty(\partial\Omega_{in})})^2}{\beta\,|Y^p|^2}+\frac{C_{72}\;||\vec{q}\cdot\vec{n}||_{L^\infty(\partial\Omega_{in})}}{|Y^p|}\right)}_{=:C_{73}}\sum_{k=1}^{I_2}\int_0^T\int_\Omega |v_{\delta_k}|^2\,dx\,dt$$
$$\leq \frac{\beta}{4}\sum_{k=1}^{I_2}\int_0^T\int_\Omega |\nabla v_{\delta_k}|^2\,dx\,dt + C_{73}\sum_{k=1}^{I_2}\int_0^T\int_\Omega |v_{\delta_k}|^2\,dx\,dt, \tag{4.2.292}$$

where C_{72} and C_{73} are independent of v_δ and δ. Therefore invoking the estimates (4.2.284), (4.2.285), (4.2.284), (4.2.287), (4.2.291) and (4.2.292) in (4.2.290) we see that the r.h.s. of (4.2.290) is finite and independent of δ, i.e.,

$$\frac{\beta}{4}\sum_{k=1}^{I_2}\int_0^T\int_\Omega |\nabla v_{\delta_k}|^2\,dx\,dt \;\leq\; C_{74},$$

i.e.,

$$|||\nabla v_\delta|||_{L^2((0,T);L^2(\Omega))^{I_2}} \;\leq\; C_{75}, \tag{4.2.293}$$

where C_{75} $(:= (\frac{4}{\beta}C_{74})^{\frac{1}{2}})$ is independent of δ.
(x) Testing (4.2.271) by $\phi \in L^2((0,T);H^{1,2}(\Omega))^{I_2}$ and following the steps of part (ix) leads us to

$$\left|\left|\left|\frac{\partial v_\delta}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}} \leq C_{76}, \tag{4.2.294}$$

where C_{76} is independent of δ. Arguments similar to steps (ix) and (x) will yield
(xi)

$$|||\nabla u_\delta|||_{L^2((0,T);L^2(\Omega))^{I_1}} \;\leq\; C_{77} \tag{4.2.295}$$

and

(xii)

$$\left|\left|\left|\frac{\partial u_\delta}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}} \leq C_{78}, \tag{4.2.296}$$

where C_{77} and C_{78} are independent of δ. Therefore combining the above estimates, we obtain

$$|||u_\delta|||_{L^2((0,T);L^2(\Omega))^{I_1}} + |||u_\delta|||_{L^\infty((0,T);L^\infty(\Omega))^{I_1}} + |||\nabla u_\delta|||_{L^2((0,T);L^2(\Omega))^{I_1}}$$
$$+\left|\left|\left|\frac{\partial u_\delta}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}} + |||v_\delta|||_{L^2((0,T);L^2(\Omega))^{I_2}} + |||v_\delta|||_{L^\infty((0,T);L^\infty(\Omega))^{I_2}}$$
$$+|||\nabla v_\delta|||_{L^2((0,T);L^2(\Omega))^{I_2}} + \left|\left|\left|\frac{\partial v_\delta}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}} + |||w_\delta|||_{L^p((0,T)\times\Omega\times\Gamma)^{I_2}}$$
$$+|||w_\delta|||_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} + \left|\left|\left|\frac{\partial w_\delta}{\partial t}\right|\right|\right|_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} + \left|\left|\left|\int_\Gamma \frac{\partial w_\delta(y)}{\partial t}\,d\sigma_y\right|\right|\right|_{L^2((0,T)\times\Omega)^{I_2}}$$
$$\leq C_{63} + \sum_{p=65}^{71} C_p + \sum_{p=75}^{78} C_p = C_{62} < \infty,$$

where C_{62} $(:= C_{63} + \sum_{p=65}^{71} C_p + \sum_{p=75}^{78} C_p)$ is independent of δ. ♦

Now we have sufficient tools to send $\delta \to 0$. We follow the idea shown in [vDP04]. Let $z_\delta \in L^\infty((0,T)\times\Omega\times\Gamma)^{I_2}$ be defined by

$$z_\delta(t,x,y) = \psi_\delta(w_\delta(t,x,y)) \quad \text{for a.e. } (t,x,y) \in (0,T)\times\Omega\times\Gamma. \tag{4.2.297}$$

Due to estimate (4.2.280), there exists a triple

$$(u,v,w) \in \mathcal{F}_2^u \times \mathcal{G}_2^v \times \mathcal{H}_2^w \tag{4.2.298}$$

such that the following convergences holds:

$$\begin{array}{lllll}
\text{(i)} & u_\delta \rightharpoonup u & \text{in} & L^2((0,T);H^{1,2}(\Omega))^{I_1}. \\
\text{(ii)} & \dfrac{\partial u_\delta}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{in} & L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}. \\
\text{(iii)} & u_\delta \to u & \text{in} & L^2((0,T);L^2(\Omega))^{I_1}. \\
\text{(iv)} & v_\delta \rightharpoonup v & \text{in} & L^2((0,T);H^{1,2}(\Omega))^{I_2}. \\
\text{(v)} & \dfrac{\partial v_\delta}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t} & \text{in} & L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}. \\
\text{(vi)} & v_\delta \to v & \text{in} & L^2((0,T);L^2(\Omega))^{I_2}. \\
\text{(vii)} & w_\delta \rightharpoonup w & \text{in} & L^2((0,T)\times\Omega\times\Gamma)^{I_2}. \\
\text{(viii)} & \dfrac{\partial w_\delta}{\partial t} \rightharpoonup \dfrac{\partial w}{\partial t} & \text{in} & L^2((0,T)\times\Omega\times\Gamma)^{I_2}. \\
\text{(ix)} & \displaystyle\int_\Gamma \frac{\partial w_\delta}{\partial t}\,d\sigma_y \rightharpoonup \int_\Gamma \frac{\partial w}{\partial t}\,d\sigma_y & \text{in} & L^2((0,T)\times\Omega)^{I_2}. \\
\text{(x)} & z_\delta \overset{w^*}{\rightharpoonup} z & \text{in} & L^\infty((0,T)\times\Omega\times\Gamma)^{I_2}.
\end{array} \tag{4.2.299}$$

Theorem 4.2.3.2. *The weak limits u and v belong to $L^\infty((0,T)\times\Omega)^{I_1}$ and $L^\infty((0,T)\times\Omega)^{I_2}$ respectively.*

Proof. Investing the knowledge of strong convergences and L^∞ - estimates of $(u_\delta)_{\delta>0}$ and $(v_\delta)_{\delta>0}$ and replicating the steps of theorem 4.1.2.2.4 will yield the proof. ♦

Theorem 4.2.3.3. *The source terms $(S_1R(u_\delta,v_\delta))_{\delta>0}$ and $(S_2R(u_\delta,v_\delta))_{\delta>0}$ are strongly convergent to $S_1R(u,v)$ and to $S_2R(u,v)$ in $L^2((0,T)\times\Omega)^{I_1}$ and $L^2((0,T)\times\Omega)^{I_2}$ respectively.*

Proof. The strong convergences of $(u_\delta)_{\delta>0}$ and $(v_\delta)_{\delta>0}$ and the L^∞ - estimates of u_δ, v_δ, u and v finish off the proof. Follow the steps of theorem 4.2.2.2.2.4. ♦

Remark 4.2.3.4. Note that the strong convergence of $(S_1R(u_\delta,v_\delta))_{\delta>0}$ in $L^2((0,T)\times\Omega)^{I_1}$ implies its strong convergence in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}$ and this shows its weak convergence in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}$. Similarly $(S_2R(u_\delta,v_\delta))_{\delta>0}$ is weakly convergent in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}$.

Theorem 4.2.3.5. *There exists a unique weak solution*

$$u \in \mathcal{F}_2^u \cap L^\infty((0,T)\times\Omega)^{I_1},\ v \in \mathcal{G}_2^v \cap L^\infty((0,T)\times\Omega)^{I_2},\ w \in \mathcal{H}_2^w \text{ and } z \in \mathcal{M}_\infty^z$$

of the problem

$$\frac{\partial u}{\partial t} - \nabla\left(P\nabla u - \frac{1}{|Y^p|}(\vec{q}-\vec{q}_0)u\right) = S_1R(u,v) \qquad \text{in } (0,T)\times\Omega, \tag{4.2.300}$$

$$-\left(P\nabla u - \frac{1}{|Y^p|}(\vec{q}-\vec{q}_0)\right)\cdot\vec{n} = d \qquad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.301}$$

$$-(P\nabla u + \frac{1}{|Y^p|}\vec{q}_0)\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.302}$$

$$u(0,x) = u_0(x) \qquad \text{in } \Omega, \tag{4.2.303}$$

$$\frac{\partial v}{\partial t} - \nabla\left(P\nabla v - \frac{1}{|Y^p|}(\vec{q}-\vec{q}_0)v\right) = S_2 R(u,v)$$
$$- \frac{1}{|Y^p|}\int_\Gamma \frac{\partial w}{\partial t}\,dy \text{ in } (0,T)\times\Omega, \tag{4.2.304}$$
$$-\left(P\nabla v - \frac{1}{|Y^p|}(\vec{q}-\vec{q}_0)v\right)\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega_{in}, \tag{4.2.305}$$
$$-(P\nabla v + \frac{1}{|Y^p|}\vec{q}_0)\cdot\vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega_{out}, \tag{4.2.306}$$
$$v(0,x) = v_0(x) \qquad \text{in } \Omega, \tag{4.2.307}$$
$$\frac{\partial w}{\partial t} = -k_d z \qquad \text{on } (0,T)\times\Omega\times\Gamma, \tag{4.2.308}$$
$$w(0,x,y) = w_0(x,y) \qquad \text{on } \Omega\times\Gamma, \tag{4.2.309}$$
$$z \in \psi(w) \qquad \text{on } (0,T)\times\Omega\times\Gamma, \tag{4.2.310}$$

where

$$\psi(w_m) = \begin{cases} \{0\} & \text{if } w_m < 0, \\ [0,1] & \text{if } w_m = 0, \\ \{1\} & \text{if } w_m > 0, \end{cases} \tag{4.2.311}$$

which satisfies the estimate

$$\begin{aligned} &|||u|||_{L^2((0,T);L^2(\Omega))^{I_1}} + |||u|||_{L^\infty((0,T);L^\infty(\Omega))^{I_1}} + |||\nabla u|||_{L^2((0,T);L^2(\Omega))^{I_1}} \\ &+ \left|\left|\left|\frac{\partial u}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}} + |||v|||_{L^2((0,T);L^2(\Omega))^{I_2}} + |||v|||_{L^\infty((0,T);L^\infty(\Omega))^{I_2}} \\ &+ |||\nabla v|||_{L^2((0,T);L^2(\Omega))^{I_2}} + \left|\left|\left|\frac{\partial v}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}} + |||w|||_{L^p((0,T)\times\Omega\times\Gamma)^{I_2}} \\ &+ |||w|||_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} + \left|\left|\left|\frac{\partial w}{\partial t}\right|\right|\right|_{L^2((0,T)\times\Omega\times\Gamma)^{I_2}} + \left|\left|\left|\int_\Gamma \frac{\partial w(y)}{\partial t}\,d\sigma_y\right|\right|\right|_{L^2((0,T)\times\Omega)^{I_2}} \\ &= C_{79} < \infty, \end{aligned} \tag{4.2.312}$$

where C_{79} is independent of δ.

Proof. The estimate (4.2.312) follows immediately from the weak convergences in (4.2.299). Moreover, (u,v,w) satisfies the equation (4.2.300)-(4.2.309). Here special attention needs to be paid to prove (4.2.310). This part is shown in theorem 2.21 in [vDP04]. ♦

Lemma 4.2.3.6. *Suppose that $p > n+2$ and $\vec{q}_0 \in L^2((0,T)\times\Omega)$. If we define a map $\Lambda_{\vec{q}_0}$: $L^\infty((0,T)\times\Omega)^{I_2} \to L^p((0,T);H^{1,q}(\Omega)^*)^{I_2}$ by*

$$\langle\Lambda_{\vec{q}_0}\phi,\zeta\rangle := \frac{1}{|Y^p|}\sum_{k=1}^{I_2}\int_0^T\int_\Omega \phi_k\vec{q}_0\cdot\nabla\zeta_k\,dx\,dt, \quad \zeta\in L^q((0,T);H^{1,q}(\Omega))^{I_2},$$

then the map $\Lambda_{\vec{q}_0}$ is well defined and continuous.

Proof. The proof is similar as the one for lemma 4.2.1.3.1. ♦

Theorem 4.2.3.7. *There exists a unique positive global weak solution*

$$u\in\mathcal{F}_p^u\cap L^\infty((0,T)\times\Omega)^{I_1},\ v\in\mathcal{G}_p^v\cap L^\infty((0,T)\times\Omega)^{I_2},\ w\in\mathcal{H}_p^w,\ z\in\mathcal{M}_\infty^z \tag{4.2.313}$$

of the problem (4.2.300)-(4.2.311).

Proof. Step 1. (a) Multiplying both sides of the m-*th* ODE of (4.2.308) by $\frac{\partial w_m}{\partial t}\left|\frac{\partial w_m}{\partial t}\right|^{p-2}$ and integrating, we obtain

$$\int_0^T\int_\Omega\int_\Gamma \frac{\partial w_m}{\partial t}\frac{\partial w_m}{\partial t}\left|\frac{\partial w_m}{\partial t}\right|^{p-2} dx\, d\sigma_y\, dt = -k_d\int_0^T\int_\Omega\int_\Gamma \psi(w_m)\frac{\partial w_m}{\partial t}\left|\frac{\partial w_m}{\partial t}\right|^{p-2} dx\, d\sigma_y\, dt,$$

i.e.,

$$\int_0^T\int_\Omega\int_\Gamma \left|\frac{\partial w_m}{\partial t}\right|^{p} dx\, d\sigma_y\, dt \le \int_0^T\int_\Omega\int_\Gamma k_d\,|\psi(w_m)|\left|\frac{\partial w_m}{\partial t}\right|^{p-1} dx\, d\sigma_y\, dt,$$

i.e.,

$$\int_0^T\int_\Omega\int_\Gamma \left|\frac{\partial w_m}{\partial t}\right|^{p} dx\, d\sigma_y\, dt \le \int_0^T\int_\Omega\int_\Gamma k_d\left|\frac{\partial w_m}{\partial t}\right|^{p-1} dx\, d\sigma_y\, dt, \text{ since } |\psi(w_m)|\le 1$$

i.e.,

$$\int_0^T\int_\Omega\int_\Gamma \left|\frac{\partial w_m}{\partial t}\right|^{p} dx\, d\sigma_y\, dt \le \int_0^T\int_\Omega\int_\Gamma \left[\frac{p-1}{p}\left|\frac{\partial w_m}{\partial t}\right|^{p} + \frac{1}{p}k_d^p\right] dx\, d\sigma_y\, dt,$$

i.e.,

$$\frac{1}{p}\int_0^T\int_\Omega\int_\Gamma \left|\frac{\partial w_m}{\partial t}\right|^{p} dx\, d\sigma_y\, dt \le \frac{k_d^p}{p}\int_0^T\int_\Omega\int_\Gamma dx\, d\sigma_y\, dt,$$

i.e.,

$$\int_0^T\int_\Omega\int_\Gamma \left|\frac{\partial w_m}{\partial t}\right|^{p} dx\, d\sigma_y\, dt \le k_d^p\, T\,|\Omega|\,|\Gamma|,$$

i.e.,

$$\sum_{m=1}^{I_2}\int_0^T\int_\Omega\int_\Gamma \left|\frac{\partial w_m}{\partial t}\right|^{p} dx\, d\sigma_y\, dt \le k_d^p\, T\,|\Omega|\,|\Gamma|\, I_2,$$

i.e.,

$$\left|\left|\left|\frac{\partial w}{\partial t}\right|\right|\right|_{L^p((0,T)\times\Omega\times\Gamma)^{I_2}} < \infty. \tag{4.2.314}$$

(b) Multiplying both side of the m-*th* ODE of (4.2.308) by $w_m\,|w_m|^{p-2}$ and integrating, we obtain

$$\int_0^t\int_\Omega\int_\Gamma w_m\,|w_m|^{p-2}\frac{\partial w_m}{\partial \theta}\, dx\, d\sigma_y\, d\theta = -k_d\int_0^t\int_\Omega\int_\Gamma \psi(w_m)\,|w_m|^{p-2}\, w_m\, dx\, d\sigma_y\, d\theta$$

i.e.,

$$\frac{1}{p}\int_0^t \frac{\partial}{\partial\theta}\,||w_m(\theta)||^p_{L^p(\Omega\times\Gamma)}\, d\theta \le \int_0^t\int_\Omega\int_\Gamma \left[\frac{p-1}{p}\,|w_m|^p + \frac{1}{p}k_d^p\right] dx\, d\sigma_y\, d\theta,$$

i.e.,

$$||w_m(t)||^p_{L^p(\Omega\times\Gamma)} \le ||w_m(0)||^p_{L^p(\Omega\times\Gamma)} + k_d^p\; T\;|\Omega|\,|\Gamma| + (p-1)\int_0^t ||w_m(\theta)||^p_{L^p(\Omega\times\Gamma)}\, d\theta.$$

Note that $||w_m(0)||_{L^p(\Omega\times\Gamma)} < \infty$ by footnote 43. A straightforward application of Gronwall's inequality and integration from 0 to T leaves

$$|||w|||_{L^p((0,T)\times\Omega\times\Gamma)^{I_2}} < \infty. \tag{4.2.315}$$

Therefore (4.2.314)-(4.2.315) shows that $w \in \mathcal{H}_p^w$.
Step 2. The abstract formulation of the problem (4.2.304)-(4.2.307) is given by

$$\frac{\partial v}{\partial t} + Av = f(v) + f_{bound}(v), \tag{4.2.316}$$
$$v(0,x) = v_0(x), \tag{4.2.317}$$

where $f(v) = S_2R(u,v) + \kappa v - \frac{1}{|Y^p|}\int_\Gamma \frac{\partial w}{\partial t}\,d\sigma_y - \vec{q}\cdot\nabla v - \Lambda_{\vec{q}_0} v$, $f_{bound}(v) = Q_{\partial\Omega_{in}}(v)$ and the operator $A : H^{1,p}(\Omega_\varepsilon^p)^I \to [H^{1,q}(\Omega_\varepsilon^p)^*]^I$ is defined as $Av_\varepsilon := (A_1v_1, A_2v_2, ..., A_{I_2}v_{I_2})$ such that for $1 \leq k \leq I_2$,

$$\begin{aligned}\langle A_k v_k, \zeta_k\rangle := &\int_\Omega P\nabla v_k(x)\cdot\nabla\zeta_k(x)\,dx\\ &+ \kappa\int_\Omega v_k(x)\zeta_k(x)\,dx \quad \text{for } v_k \in H^{1,p}(\Omega) \text{ and } \zeta_k \in H^{1,q}(\Omega),\end{aligned} \tag{4.2.318}$$

where $\kappa > 0$. The estimate (4.2.312) and lemma 4.2.3.6 imply $f \in L^p((0,T);H^{1,q}(\Omega)^*)^{I_2}$. From lemma 4.2.1.3.1 it follows that $f_{bound} \in L^p((0,T);H^{1,q}(\Omega)^*)^{I_2}$. Moreover the initial condition $v_0 \in \mathcal{X}_p^v$. Therefore by theorem 3.3.1, there exists a unique solution $v \in \mathcal{G}_p^v$ of (4.2.316)-(4.2.317), i.e., a unique solution of (4.2.304)-(4.2.307) and it satisfies the estimate

$$|||v|||_{\mathcal{G}_p^v} \leq C_{80}\left(|||v_0|||_{\mathcal{X}_p^v} + |||f + f_{bound}|||_{L^p((0,T);H^{1,q}(\Omega)^*)^{I_2}}\right). \tag{4.2.319}$$

Step 3.: Again using the estimate (4.2.312) and following the arguments of step 2, we obtain the existence of a unique weak solution $u \in \mathcal{F}_p^u$ of (4.2.300)-(4.2.303) This completes the proof. ♦

Chapter 5

Numerical Simulations

In this chapter, the models M1 and M2 are investigated numerically. For the sake of illustration, we restrict ourselves to relatively simple 2-dimensional situations. For the numerical simulations, COMSOL Multiphysics 4.3a (see [Com10]) is used. In section 5.1 model M1 and in section 5.2 model M2 are examined. For both the models, we solve the micro problem, the cell-problems and the macro problem respectively. The scaling parameter ε and the regularization parameter δ are chosen as 0.2 and 0.001 respectively. All the figures in this chapter are generated by the author using COMSOL Multiphysics 4.3a.

5.1 Simulation of Model M1

The model M1 at the micro scale (see section 2.5.2) is given by

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} - \nabla\cdot D\nabla u_\varepsilon &= SR(u_\varepsilon) &&\text{in} \quad (0,T)\times\Omega_\varepsilon^p, && (5.1.1)\\
u_\varepsilon(0,x) &= u_0(x) &&\text{in} \quad \Omega_\varepsilon^p, && (5.1.2)\\
-D\nabla u_\varepsilon\cdot\vec{n} &= 0 &&\text{on} \quad (0,T)\times\partial\Omega, && (5.1.3)\\
-D\nabla u_\varepsilon\cdot\vec{n} &= 0 &&\text{on} \quad (0,T)\times\Gamma_\varepsilon. && (5.1.4)
\end{aligned}$$

The homogenized form of (5.1.1)-(5.1.4) is given by

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nabla\cdot P\,\nabla u &= SR(u) &&\text{in} \quad (0,T)\times\Omega, && (5.1.5)\\
-P\,\nabla u\cdot\vec{n} &= 0 &&\text{on} \quad (0,T)\times\partial\Omega, && (5.1.6)\\
u(0,x) &= u_0(x) &&\text{in} \quad \Omega. && (5.1.7)
\end{aligned}$$

Here $P = (p_{jk})_{1\le j,k\le 2}$ is a second order tensor with components

$$p_{jk} = \int_{Y^p} \frac{D}{|Y^p|}\left(\delta_{jk} + \frac{\partial a_j}{\partial y_k}\right) dy \text{ for all } j,k = 1,2, \tag{5.1.8}$$

where for all $j = 1,2$, (a_j) is the solution of the cell-problem

$$\begin{aligned}
-\nabla_y\cdot(D(\nabla_y a_j(y) + e_j)) &= 0 &&\text{for } y\in Y^p, && (5.1.9)\\
-D(\nabla_y a_j(y) + e_j)\cdot\vec{n} &= 0 &&\text{for } y\in\Gamma, && (5.1.10)\\
y\mapsto a_j(y) &\text{ is } Y-\text{periodic}. && && (5.1.11)
\end{aligned}$$

The physics setting: Let us consider a domain $\Omega := [0,1.2]\times[0,1]$ in $\mathbb{R}^2$. Assume that $Y = [0,1]\times[0,1]\subset\mathbb{R}^2$ is the representative cell with $Y^s := B((0.5,0.5),0.15)$ as the solid inclusion[46]. Suppose that four mobile species A, B, M and N are present inside Ω. The chemical species diffuse and react with each other (cf. figure 5.1.1).

[46] For $r\in\mathbb{R}^n$, $B(r,\epsilon)$ denotes an open ball centered at r and radius ϵ.

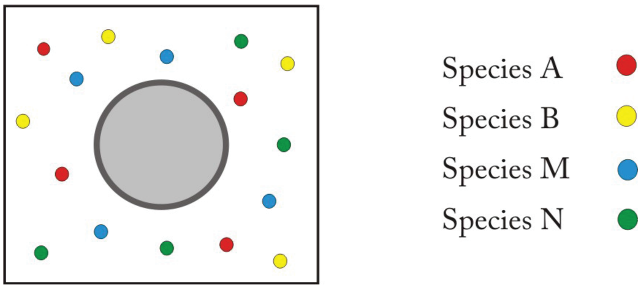


Figure 5.1.1: Diffusion-reaction of species A, B, M and N.

The reaction is reversible and is given by

$$2A + 3B \rightleftharpoons M + 2N. \tag{5.1.12}$$

The stoichiometric coefficients are -2, -3, 1 and 2, and the reaction rates for each species can be given by (2.4.7). Here $I = 4$ and $J = 1$.

5.1.1 Simulation at the Micro Scale

Let u_{ε_i} denote the concentration of *i-th* species for $1 \leq i \leq 4$. We choose the scaling parameter $\varepsilon = 0.2$. Also, let $D = 1.0$, $k_j^f = 1.8$, $k_j^b = 12.2$. Initially, let us assume $u_{\varepsilon_1}(0,x) = 5x$, $u_{\varepsilon_2}(0,x) = 2(x+3)$, $u_{\varepsilon_3}(0,x) = 5x$ and $u_{\varepsilon_4}(0,x) = 2x$. We "choose coarser option" mesh available in COMSOL to discretize the domain Ω_ε^p. The triangulization of the domain Ω_ε^p is depicted in the following figure:

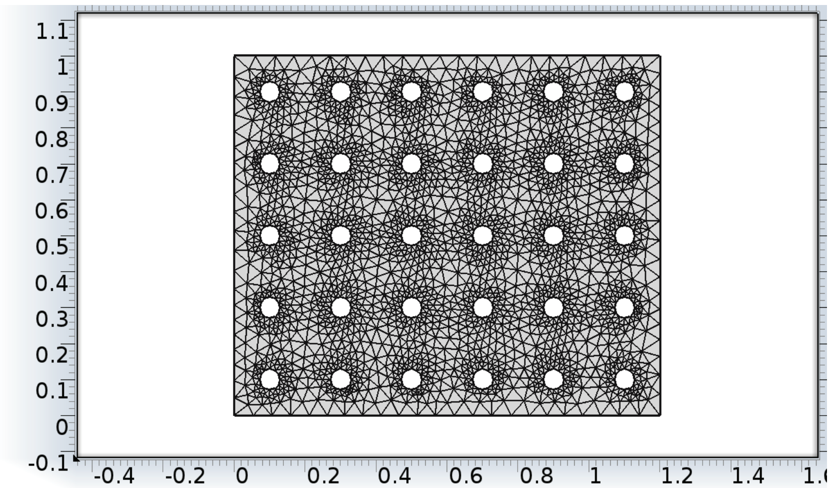


Figure 5.1.2: The triangulization of Ω_ε^p for $\varepsilon = 0.2$.

We solved the system of diffusion-reaction equations at the micro scale for $t = 10$ secs. We notice: the number of elements for mesh = 4930, the number of degrees of freedom = 10640 and the time taken by the solver = 104 secs. However, here we compare the solution for species A only at the micro and the macro scale, since the comparison of the solutions for the rest of the species can be done analogously. The concentration of species A is depicted in the following pictures for $t = 0.25$ secs, $t = 0.35$ secs, $t = 3$ secs and $t = 10$ secs respectively:

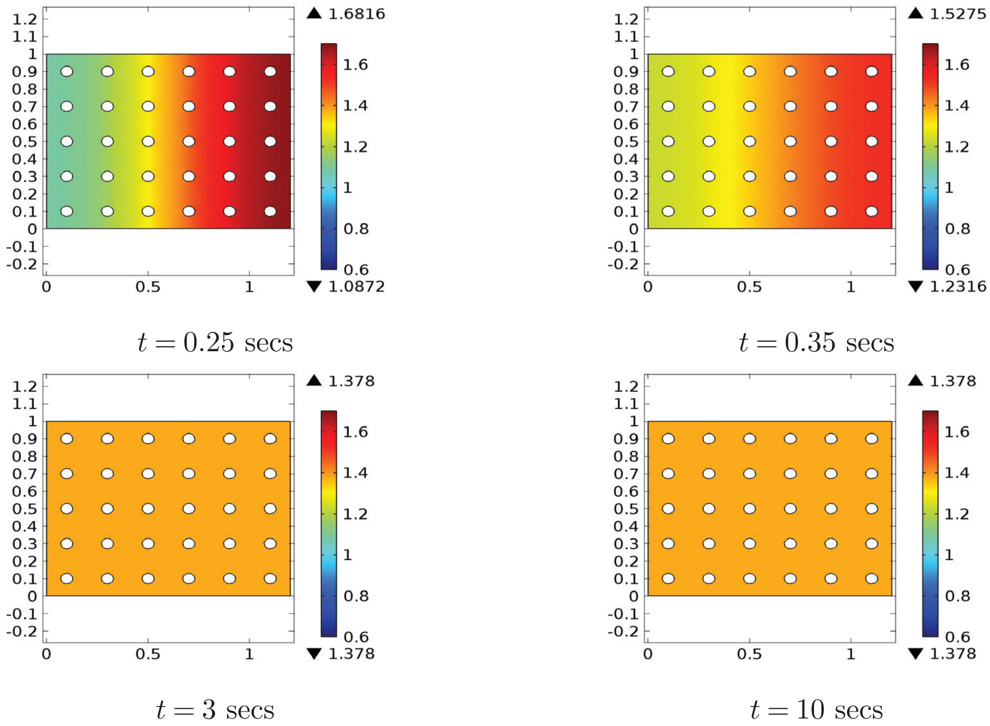


Figure 5.1.3: Concentration of species A in Ω_ε^p at different time.

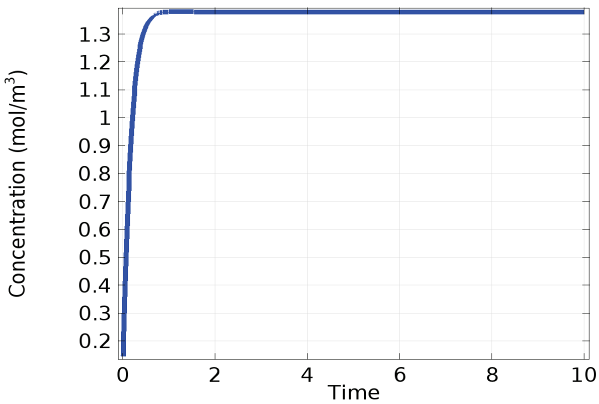


Figure 5.1.4: Concentration of species A at the top left point of Ω_ε^p in 10 secs.

In figure 5.1.3, we see the change in concentration of species A at different time. As time progesses, the concentration of species A increases and due to reversible reaction after $t = 3$ secs, the reaction reaches equilibrium. This is also shown in the figure 5.1.4 where the concentration of species A at a point (top left point of the domain) is plotted. Now we compute the effective diffusion tensor for species A. We commence by solving the cell-problems (5.1.9)-(5.1.11) in Y.

5.1.2 Solution of the Cell-Problems

We choose the "finer mesh option" (available in COMSOL) for the triangulization of the cell Y. The triangulization of Y is depicted below:

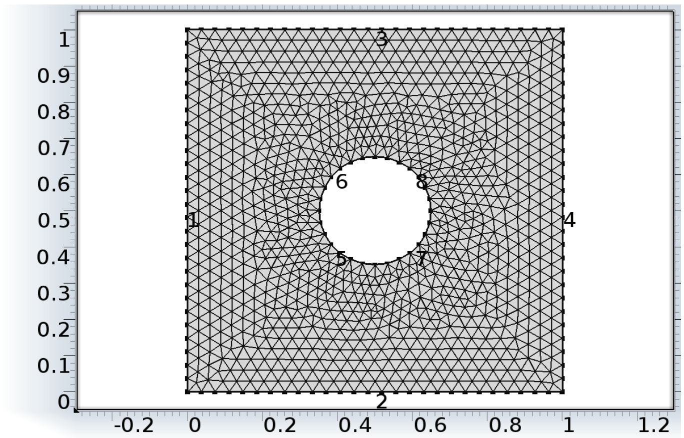


Figure 5.1.5: The triangulization of cell Y.

In the following figure, we see the solutions of the cell-problems.

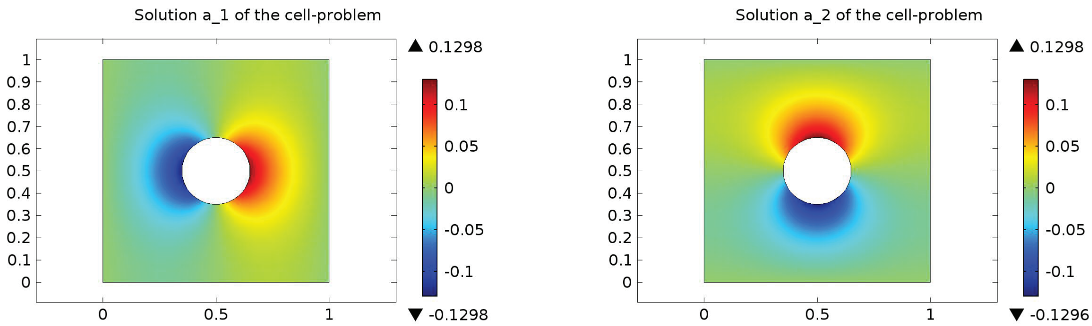


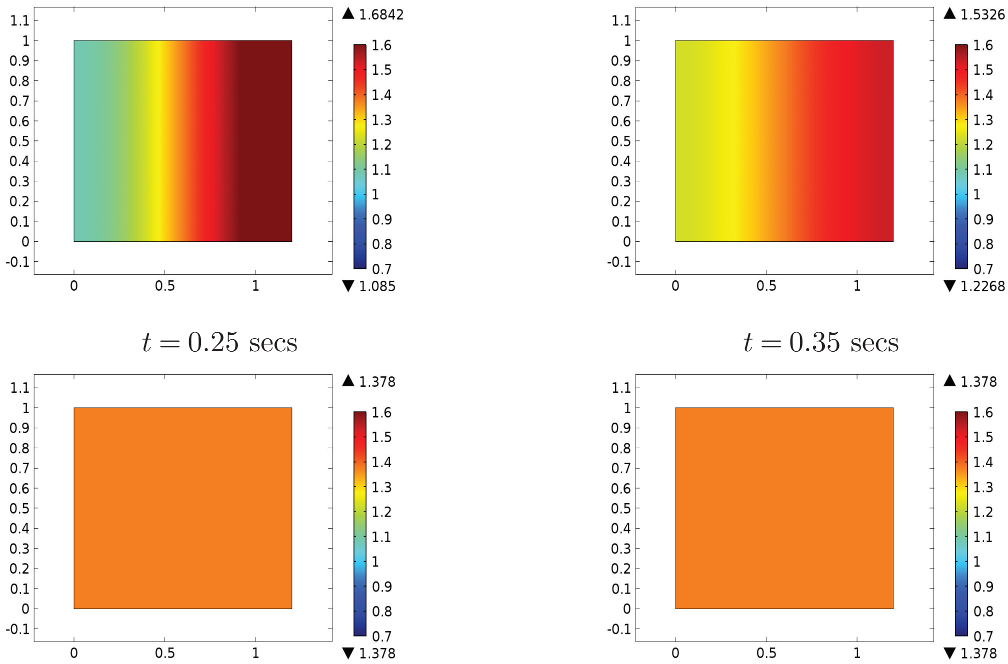
Figure 5.1.6: Solution a_j of the cell-problem for $j = 1, 2$.

With the help of 'Derived Values' feature in COMSOL, we compute the diffusive tensor by the formula (5.1.8). Thus we obtain

$$P = (p_{jk})_{\substack{1\le j\le 2\\ 1\le k\le 2}} = \begin{bmatrix} 0.93409 & 4.19\times 10^{-7} \\ 4.19\times 10^{-7} & 0.93409 \end{bmatrix}. \tag{5.1.13}$$

5.1.3 Simulation at the Macro Scale

For the simulation of upscaled model, we choose P from (5.1.13), $k_j^f = 1.8$, $k_j^b = 12.2$. Initially, $u_1(0,x) = 5x$, $u_2(0,x) = 2(x+3)$, $u_3(0,x) = 5x$ and $u_4(0,x) = 2x$. We choose the coarser mesh (in COMSOL) for Ω with 144 elements. We also notice that: the number of degrees of freedom = 352 and the time taken by the solver = 11 secs. The numerical simulation is shown in the following pictures:



$t = 0.25$ secs $t = 0.35$ secs

$t = 2.4$ secs $t = 10$ secs

Figure 5.1.7: Concentration of species A in Ω at different time scales.

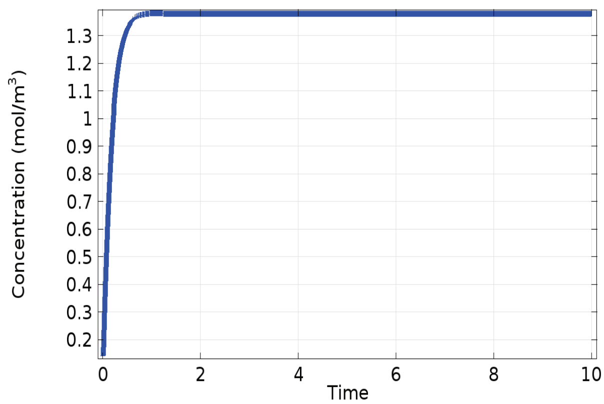


Figure 5.1.8: Concentration of species A at the top left point of Ω in 10 secs.

Conclusions: Firstly, we notice that for the same type of mesh the solver takes less time to solve the macro problem than to solve the micro problem. Therefore the upscaled model is computationally efficient. Secondly, the upscaled model gives us the global information of the properties related to our porous medium. In figure 5.1.7, it is shown that as time progresses there is an increase in the concentration of species A and after $t = 2.4$ secs the reaction reaches equilibrium as expected. By comparing the figures 5.1.3 and 5.1.7, we can notice that the upscaled model (5.1.5)-(5.1.7) is a good approximation to our original micro problem (5.1.1)-(5.1.4). This can also be seen by comparing the figures 5.1.4 and 5.1.8.

5.2 Simulation of Model M2

In this section, we replicate the process of section 5.1 for model M2. The micro and the macro problems are given by (2.5.21)-(2.5.36) and (4.2.300)-(4.2.311) respectively.

The physics setting: Let Ω, Y, Y^s and Γ be like in section 5.1. Suppose that a chemical species A is present in the fluid which enters in domain Ω. The dissolution of immobile species (present on the surface of the solid parts) occurs on Γ. An another mobile chemical species B is supplied via dissolution. The mobile species A and B react under the following reversible reaction (see also figure 5.2.1):

$$2A \rightleftharpoons 3B. \tag{5.2.1}$$

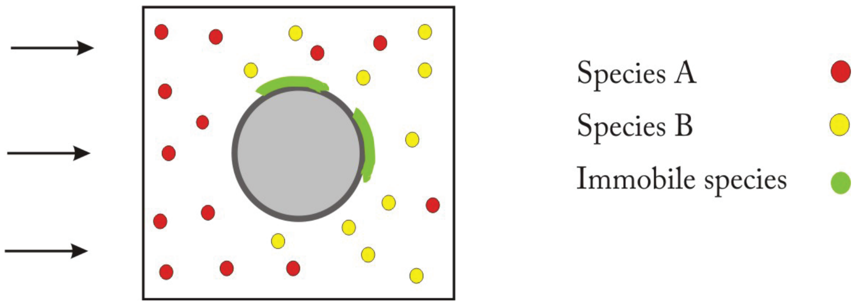


Figure 5.2.1: Presence of mobile species A and B in the pore space and immobile species on Γ.

The stoichiometric coefficients of A and B are -2 and 3 respectively and the reaction rates of these species can be given by (2.4.7). By choosing an appropriate $\vec{q}_\varepsilon$ which satisfies (2.5.36), the numerical simulations for mobile and immobile species (both at the micro and the macro scale) can be conducted and the results can be compared as we did in section 5.1.

Chapter 6

Summary and Outlook

6.1 Summary

In chapter 4, we proved the positivity, existence and uniqueness of the global solution for the models I and II respectively. At first, both the models are considered at the micro scale. Model M1 is considered without advection. In section 4.1, we proved the existence of a unique positive global weak solution of model M1. In section 4.2, we showed the existence of a unique positve global weak solution for model M2. We considered a complex scenario by incorporating dissolution in model M2. In order to prove the existence of the solution for both models, with the help of a Lyapunov functional, we obtained inequalities like (4.1.34), (4.2.78) and (4.2.152). These inequalities gave us *global a-priori estimates* of the solution. The inequalities of this type can also be found in the works of Glitzky, Gröger and Hünlich (cf. theorems 3.1 and 3.2 in [GGH92]) to solve nonlinear parabolic equations. These inequalites are proved to be a very efficient tool in order to show the existence of the global solution. After proving the existence of the solution, we upscaled the models (M1 and M2) from the micro to the macro scale using *two-scale convergence* and *periodic unfolding*. The homogenization (upscaling) of models M1 and M2 are shown in sections 4.1.2 and 4.2.2 respectively. We performed the numerical computations at the micro scale and at the macro scale for both the models in sections 5.1 and 5.2 respectively. From the conclusions of sections 5.1 and 5.2, we see that the upscaled models for M1 and M2 at the macro scale are a good approximation to the models M1 and M2 considered at the micro scale. For the future work, following continuations can be made.

6.2 Outlook

- **Use of different types of scaling:** For the sake of simplicity, in this work natural scale has been chosen. It is already shown in the works of Peter and Böhm (cf. [PB08], [PB05], see also [Pet06]) that with different choices of scaling factor at the micro scale one obtains different types of upscaled models at the macro scale. Thus it would be very interesting to have a different scaling in (2.5.16)-(2.5.19) or in (2.5.21)-(2.5.35). The idea to use other types of scaling can be motivated from: how much our porous medium is perforated ? or, how do the parameters (e.g. diffusion coefficient) involved in the equations oscillate ? or, what kind of flux conditions are needed on the surface of the solid parts ? etc. For a brief explaination see [All92]. Also see [PB08], [NR92], [Dob12], [Fre11] and references therein.

- **Different types of diffusion coefficients:** The following generalizations can be made for the diffusion coefficients:

  ▶ In this work, we considered the same diffusion coefficient for all the mobile species. We required this assumption to establish the inequality (4.1.49). Pierre has proved the existence of the global solution of a parabolic system with two different diffusion coefficients but, to our knowledge, the existence results for the global solution in case of I (> 2) different types of diffusion coefficients is still unknown. It would be captivating to prove the existence of the global solution of the systems considered in this work for I (> 2) types of diffusion coefficients.

- ▶ We also considered constant diffusion coefficient in both models. In several real world problems the diffusion coefficients are piecewise continuous or in $L^\infty(\Omega)$, in such cases we may need to impose some condition on p. The problems with $D \in L^\infty(\Omega)$ are addressed in the works of Rehberg and Dintelmann and references therein (cf. [RDR09]). One can consider essentially bounded diffusion coefficients in the equations proposed in this work.

- **Lipschitz domains:** Our analysis is predicated to the domains which has sufficiently smooth boundaries but many real world problems involve Lipschitz domains. Rehberg and Dintelmann have considered such type of problems in [RDR09] (see also the references therein). Interested readers should conduct the investigations of chapter 4 for Lipschitz domains.

- **Different boundary conditions:** Following generalizations can be made for the boundary conditions:

  - ▶ The construction of Lyapunov functional also depends on the boundary conditions. In section 3.4 in [Krä11], Kräutle has indicated how one can construct the Lyapunov functional in the presence of Dirichlet BCs. Thus one can try to obtain the existence of solution in the presence of Dirichlet BCs in $H^{1,p}$ - setting.
  - ▶ To have nonlinear *inflow-outflow* boundary conditions.
  - ▶ The problems incorporated with mixed BCs, i.e., both Dirichlet and Neumann BCs has drawn a great attention of mathematician as they fit perfectly to many real world situations (see [RDR09] and references therein). In the literature, it is shown that due to the presence of mixed BCs we loose the regularity on p, i.e., $p \leq 4$. Therefore it would be very interesting to incorporate our systems with mixed type of BCs and obtain the existence of the global solution.

- **Including precipitation in the model:** Both precipitation and dissolution are widely explored in the fields chemical engineering, pharmaceutical industry and several others. In our work, we paid attention to dissolution process only, however, considering precipitation of immobile species (crystals) on the surface of the solid parts is definitely worth to inspect. See the works of Knabner, Duijn, Noorden, Böhm, Peter, Muntean and references therein (cf. [Kna86], [KvDH95], [vDP04], [vNP08], [vN09b], [vN09a], [PB08], [MB09b], [BJDR98], [FM12] etc.) for a detailed overview.

- **Moving boundary and variable geometry:** The dissolution and precipitation inside a porous medium can also lead to the problems with moving boundary. Since these processes occur on the surface of the solid parts (see figure 6.2.1), they may affect the size of the matrices and this could lead to the change in geometry of the domain (cf. [vN09a], [Pet06]).

Figure 6.2.1: The dissolution of tablet in water is taking place on the surface.[47]

In our work the boundary Γ is considered as fixed but one can consider the model proposed in this work with moving boundary Γ, i.e., Stefan like problem (cf. [vN09b], [vN09a], [vNP08], [Pet06]).

- **Including diffusion-reaction on the surface Γ:** Other than dissolution and precipitation, one can consider diffusion and reaction of chemical species on the surface of the solid parts. Such type of models has been studied in [HJ91], [NR92], [Pet03], [Dob12] etc for the linear reaction rates on Γ but one can modify such models by incorporating nonlinear reaction rates on Γ. To our knowledge the existence of the global solution and homogenization of such models are still open.

[47] This picture is taken from Qualichemlab.com.

Appendices

A. Inequalities

Here we state some elementary inequalities which we have used frequently throughout this work. The proofs of all these inequalities can be found in the appendix B.2 of [Eva98].

Lemma A.1 (Young's inequality with ϵ). *Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that ϵ, a and $b > 0$, then*

$$ab \leq \epsilon a^p + C(\epsilon) b^q, \tag{A.1}$$

where $C(\epsilon) = (\epsilon p)^{-\frac{q}{p}} q^{-1}$.

Lemma A.2 (Hölder's inequality). *Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose further that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then*

$$||uv||_{L^1(\Omega)} \leq ||u||_{L^p(\Omega)} \, ||v||_{L^q(\Omega)} \, . \tag{A.2}$$

Lemma A.3 (Generalized Hölder's inequality). *Let $1 \leq p_1, p_2, ..., p_r \leq \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_r} = 1$. Assume that $u_s \in L^{p_s}(\Omega)$ for s=1,2,..., r, then*

$$||u_1 u_2 ... u_r||_{L^1(\Omega)} \leq \prod_{s=1}^{r} ||u_s||_{L^{p_s}(\Omega)} \, . \tag{A.3}$$

Lemma A.4 (Minkowski's inequality). *Let $1 \leq p \leq \infty$ and $u, v \in L^p(\Omega)$, then*

$$||u + v||_{L^p(\Omega)} \leq ||u||_{L^p(\Omega)} + ||v||_{L^p(\Omega)} \, . \tag{A.4}$$

Lemma A.5 (Generalized Minkowski's inequality). *Let $1 \leq p \leq \infty$ and $u_s \in L^p(\Omega)$ for $s = 1, 2, ..., r$, then*

$$\left|\left| \sum_{s=1}^{r} u_s \right|\right|_{L^p(\Omega)} \leq \sum_{s=1}^{r} ||u_s||_{L^p(\Omega)} \, . \tag{A.5}$$

Lemma A.6 (Lyapunov's interpolation inequality). *Let $1 \leq p \leq q \leq r \leq \infty$ and $0 < \theta < 1$ be such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$. Assume also that $u \in L^p(\Omega) \cap L^r(\Omega)$. Then $u \in L^q(\Omega)$ and satisfies*

$$||u||_{L^q(\Omega)} \leq ||u||^{\theta}_{L^p(\Omega)} \, ||u||^{1-\theta}_{L^r(\Omega)} \, . \tag{A.6}$$

Lemma A.7 (Gronwall's inequality in differential form). *Let $u(.)$ be a nonnegative, absolutely continuous function on $[0, T]$ which satisfies the following inequality*

$$\frac{\partial u(t)}{\partial t} \leq \phi(t) u(t) + \psi(t) \qquad \text{for a.e. } t, \tag{A.7}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$u(t) \leq e^{\int_0^t \phi(s)\, ds} \left[u(0) + \int_0^t \psi(s)\, ds \right], \text{ for all } t \in [0, T]. \tag{A.8}$$

In particular, if $\frac{\partial u(t)}{\partial t} \leq \phi(t) u(t)$ and u(0)=0, then $u(t) = 0$ on $[0, T]$.

Lemma A.8 (Gronwall's inequality in integral form). *Let C_1 and $C_2 \geq 0$. Assume that $v(.)$ is a nonnegative, summable function on $[0,T]$ which satisfies the following integral inequality*

$$v(t) \leq C_1 \int_0^t v(s)\,ds + C_2 \quad \text{for a.e. } t. \tag{A.9}$$

Then

$$v(t) \leq C_2 \left(1 + tC_1 e^{C_1 t}\right) \text{ for a.e. } t \in [0,T]. \tag{A.10}$$

In particular, if $v(t) \leq C_1 \int_0^t v(s)\,ds$ for a.e. t, then $v(t) = 0$ a.e. $t \in [0,T]$.

Lemma A.9 (Discrete Hölder's inequality). *Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $a_k \geq 0$, $b_k \geq 0$ for $k = 1, 2, \ldots, n$, then the following inequality holds:*

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}. \tag{A.11}$$

B. Some Important Theorems and Lemmas

Theorem B.1 (Schaefer's fixed point theorem). *Let Ξ be a Banach space. Assume that $Z : \Xi \to \Xi$ is a continuous and compact map. Suppose further that the set*

$$\{u \in \Xi : u = \lambda Z(u) \text{ for some } 0 \leq \lambda \leq 1\} \tag{B.1}$$

is bounded. Then Z has a fixed point.

Proof. See theorem 4 in section 9.2.2 in [Eva98]. ♦

Lemma B.2. *For $l = 1, 2, \ldots, s$, let a_l and $\bar{a}_l \in \mathbb{R}$, then*

$$a_1 a_2 \ldots a_s - \bar{a}_1 \bar{a}_2 \ldots \bar{a}_s = \sum_{l=1}^s a_1 \ldots a_{l-1}(a_l - \bar{a}_l)\bar{a}_{l+1} \ldots \bar{a}_s. \tag{B.2}$$

Proof. It is a mere calculation. ♦

Theorem B.3 (Sobolev continuous embedding theorem). *Let $1 \leq p \leq \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$.*
(i) If $s_1 < s_2$, where s_1 and s_2 are any two positive real number, then

$$H^{s_2,p}(\Omega) \hookrightarrow H^{s_1,p}(\Omega). \tag{B.3}$$

(ii) If $1 \leq p_1 \leq p_2 \leq \infty$ and $s \in \mathbb{R}$, then

$$H^{s,p_2}(\Omega) \hookrightarrow H^{s,p_1}(\Omega). \tag{B.4}$$

(iii) If k be any positive integer, then

$$H^{k,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & 1 \leq q \leq p^* = \dfrac{np}{n-kp}, \quad kp < n, \\ L^q(\Omega), & 1 \leq q < \infty, \quad kp = n, \\ C^\alpha(\bar{\Omega}), & 0 < \alpha \leq 1 - \dfrac{n}{kp}, \quad kp > n. \end{cases} \tag{B.5}$$

Proof. (i) See theorem 6.2.3 in [BL76].
(ii) Follows directly from $L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega)$, if $p_1 \leq p_2$.
(iii) See corollary 1.3.1 [WYW06]. See also [AF03], [Eva98]. ♦

Theorem B.4 (Sobolev compact embedding theorem). *Let $1 \le p \le \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Then*

$$H^{1,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q(\Omega), & 1 \le q \le p^* < \dfrac{np}{n-p}, \quad p < n, \\ L^q(\Omega), & 1 \le q < \infty, \quad p = n, \\ C^\alpha(\bar{\Omega}), & 0 < \alpha < 1 - \dfrac{n}{p}, \quad p > n. \end{cases} \tag{B.6}$$

Proof. Cf. theorem 1.3.3 in [WYW06]. See also [AF03], [Eva98]. ♦

Theorem B.5 (Trace theorem). *Let $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Then there exists a bounded linear operator $T : H^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

$$(i)\; Tu := u|_{\partial\Omega} \quad \text{if } u \in H^{1,p}(\Omega) \cap C(\bar{\Omega})$$

and

$$(ii)\; ||Tu||_{L^p(\partial\Omega)} \le C\,||u||_{H^{1,p}(\Omega)}\,, \text{ for each } u \in H^{1,p}(\Omega), \tag{B.7}$$

where C depends on p and Ω but it is independent of u.

Proof. See theorem 1 in section 5.5 in [Eva98]. ♦

Theorem B.6 (Extension theorem). *Let $1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Suppose that V is a bounded open set such that $\Omega \subset V$. Then there exists a bounded linear operator $E : H^{1,p}(\Omega) \to H^{1,p}(\mathbb{R}^n)$ such that for each $u \in H^{1,p}(\Omega)$:*

$$(i)\; Eu := u \text{ a.e. in } \Omega,$$
$$(ii)\; Eu \text{ has a support in } V,$$

and

$$(iii)\; ||Eu||_{H^{1,p}(\mathbb{R}^n)} \le C\,||u||_{H^{1,p}(\Omega)}\,, \tag{B.8}$$

where C depends only on p, Ω and V.

Proof. See theorem 1 in section 5.4 in [Eva98]. ♦

Theorem B.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, then for all $u \in H^{1,2}(\Omega)$ the following estimate hold:*

$$||u||^2_{L^2(\partial\Omega)} \le C\,||u||_{H^{1,2}(\Omega)}\,||u||_{L^2(\Omega)}\,, \tag{B.9}$$

where the constant C is independent of u.

Proof. See lemma 5.6 in [Krä08]. ♦

Lemma B.8. *Let μ^0 be given as in (4.1.19). Then $\langle \mu^0 + \log u_\varepsilon,\, SR(u_\varepsilon)\rangle_I \le 0$.*

Proof. See pages 71 - 72 in [Krä08]. ♦

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The transport process in a porous medium is a complex phenomena. In this thesis, the heterogeneities inside a porous medium are assumed to be periodically distributed and diffusion-reaction of a finite number of chemical species are investigated. Two different models are proposed in this work. In model M1, diffusion-reaction of mobile chemical species are considered. The chemical processes are modeled via mass action kinetics and the modeling leads to a system of multi-species diffusion-reaction equations (nonlinear partial differential equations) at the micro scale. For this system of equations, existence of a unique positive global weak solution is proved by the help of a *Lyapunov functional* and *Schaefer's fixed point theorem*. The upscaled model of this system is obtained using *periodic homogenization* which is an averaging method.

In model M2, we consider diffusion-advection-reaction of two different types of mobile species (type I and type II). The type II species are supplied via dissolution process due to the presence of immobile species on the surface of the solid parts. The presence of mobile and the immobile species make the model complex and the modeling yields a coupled system of nonlinear partial differential equations. The existence of a unique positive global weak solution of this coupled system is shown. Finally, with the help of periodic homogenization, model M2 is upscaled from the micro scale to the macro scale.

Numerical simulations are conducted for both models separately. For the purpose of illustration, we restrict ourselves to relatively simple 2-dimensional situations. For models M1 and M2, simulation results at the micro scale and at the macro scale are compared.